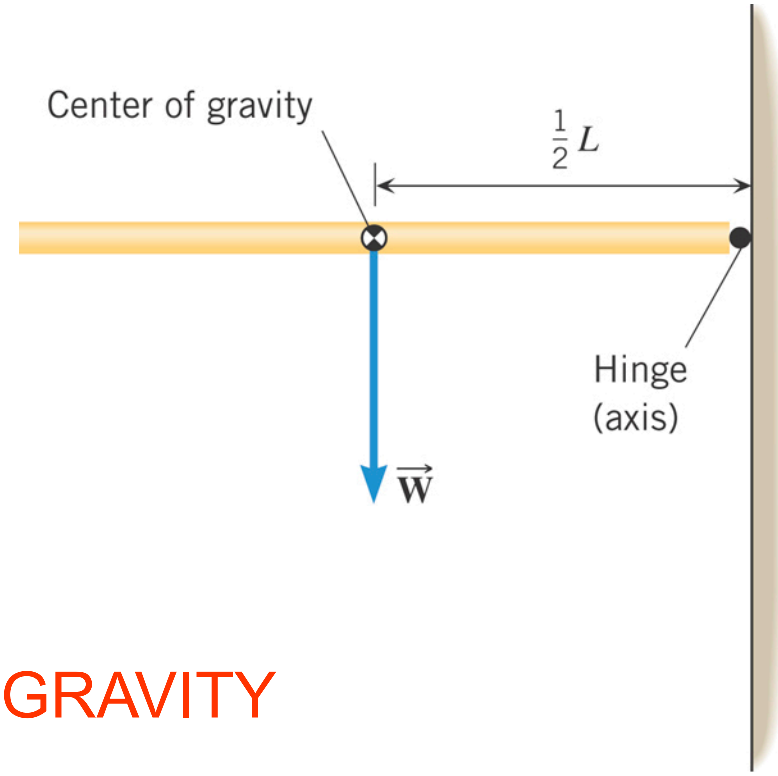
Chapter 8

Rotational Dynamics

continued

8.7 Center of Gravity

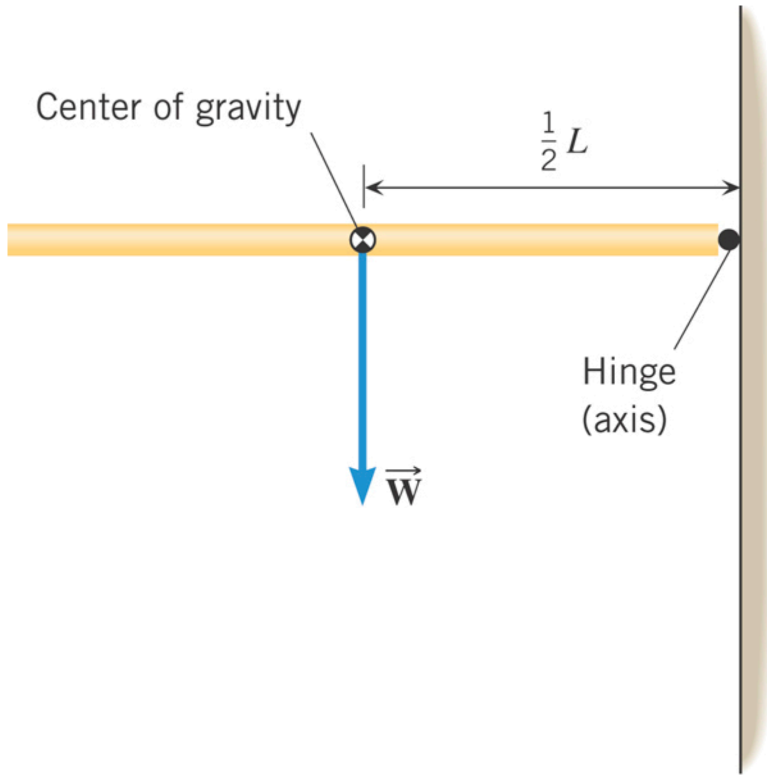


DEFINITION OF CENTER OF GRAVITY

The center of gravity of a rigid body is the point at which its weight can be considered to act when the torque due to the weight is being calculated.

8.7 Center of Gravity

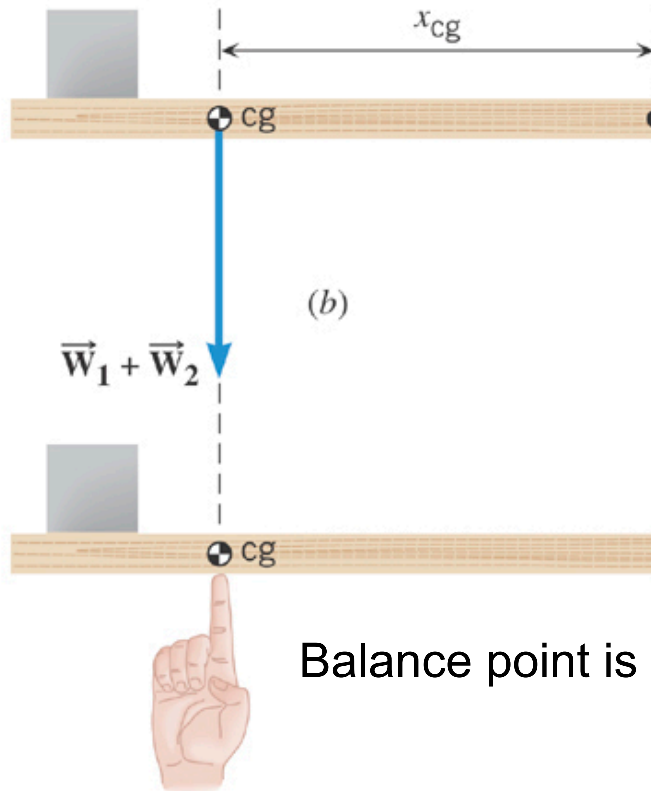
When an object has a symmetrical shape and its weight is distributed uniformly, the center of gravity lies at its geometrical center.



8.7 Center of Gravity

General Form of x_{cg}

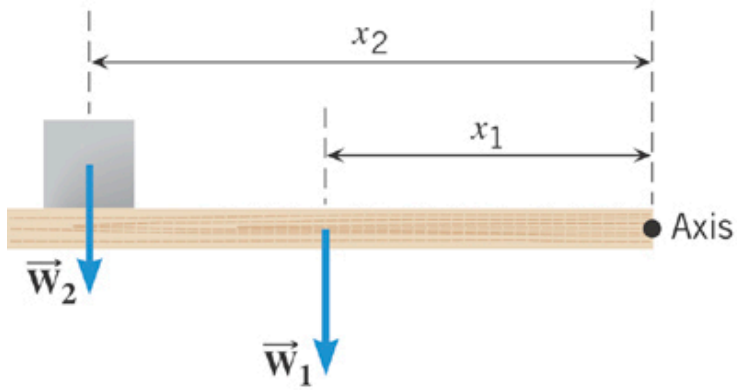
$$x_{cg} = \frac{W_1 x_1 + W_2 x_2 + \cdots}{W_1 + W_2 + \cdots}$$



(b)

(c)

Balance point is under x_{cg}



(a)

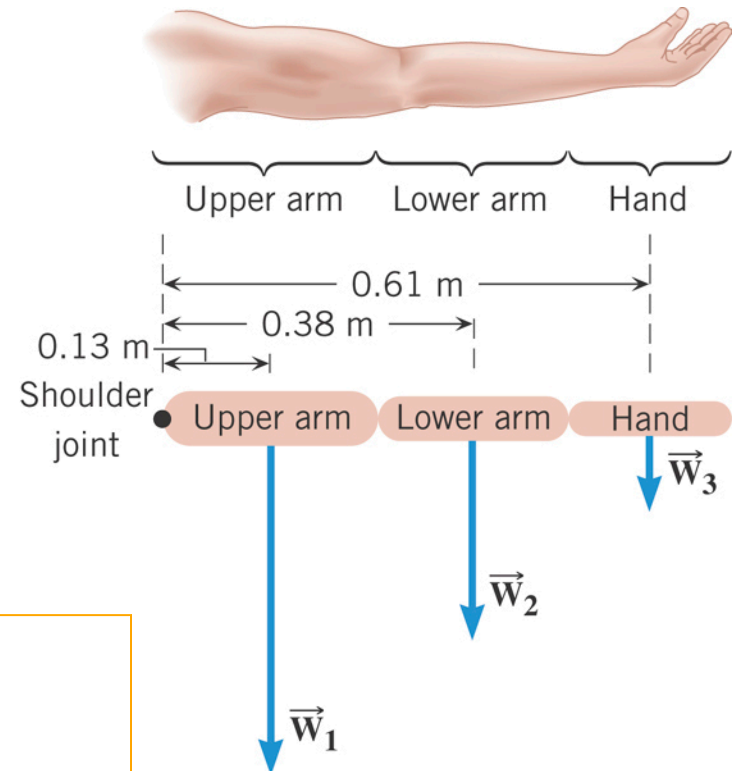
Center of Gravity, x_{cg} , for 2 masses

$$x_{cg} = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2}$$

Example: The Center of Gravity of an Arm

The horizontal arm is composed of three parts: the upper arm (17 N), the lower arm (11 N), and the hand (4.2 N).

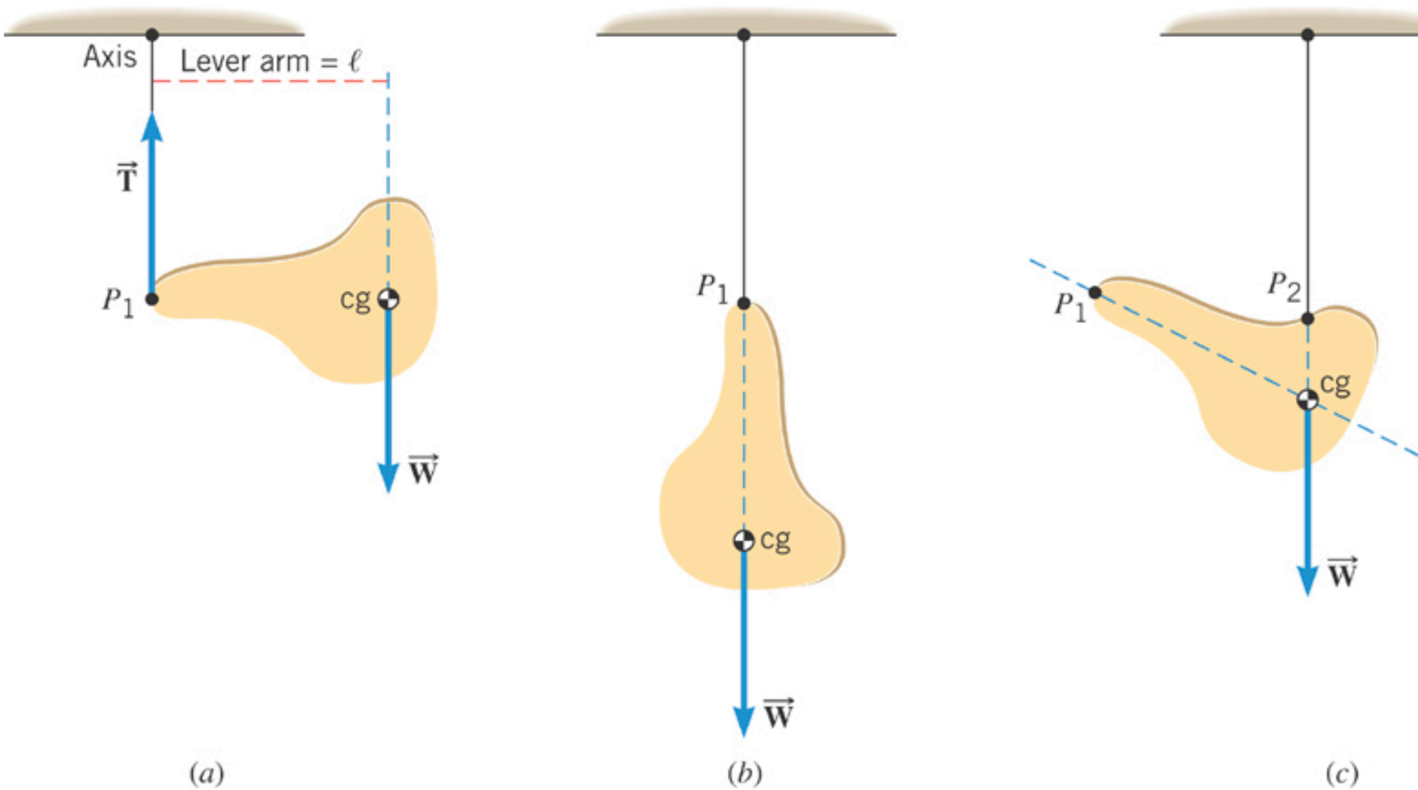
Find the center of gravity of the arm relative to the shoulder joint.



$$x_{cg} = \frac{W_1 x_1 + W_2 x_2 + W_3 x_3}{W_1 + W_2 + W_3}$$

$$= \frac{\left[17(0.13) + 11(0.38) + 4.2(0.61)\right]\text{N}\cdot\text{m}}{(17 + 11 + 4.2)\ \text{N}} = 0.28\ \text{m}$$

8.7 Center of Gravity



Finding the center of gravity of an irregular shape.

EQUILIBRIUM OF A RIGID BODY

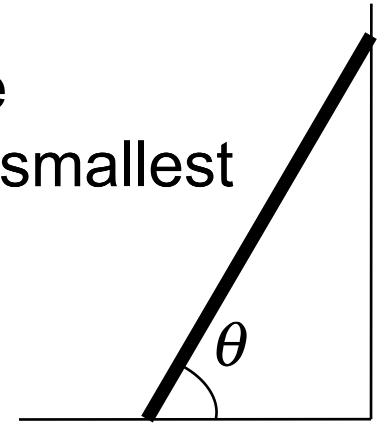
A rigid body is in equilibrium if it has zero translational acceleration and zero angular acceleration. In equilibrium, the sum of the externally applied forces is zero, and the sum of the externally applied torques is zero.

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau = 0$$

Note: constant linear speed or constant rotational speed are allowed for an object in equilibrium.

8.7 Rigid Objects in Equilibrium

Example: A board length L lies against a wall. The coefficient of friction with ground is 0.650. What is smallest angle the board can be placed without slipping?



1. Determine the forces acting on the board.

8.7 *Rigid Objects in Equilibrium*

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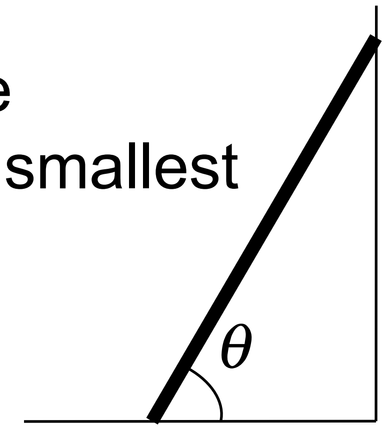


1. Determine the forces acting on the board.

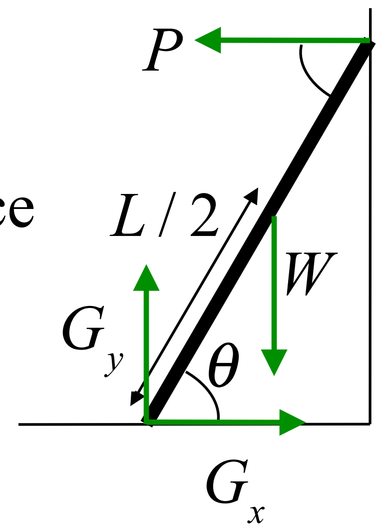
<u>Forces</u>

G_y ground normal force

G_x ground static frictional force

P wall normal force

W gravitational force



Forces:

$$G_y = W$$

$$P = G_x = \mu G_y = \mu W$$

2. Choose pivot point at ground.

8.7 *Rigid Objects in Equilibrium*

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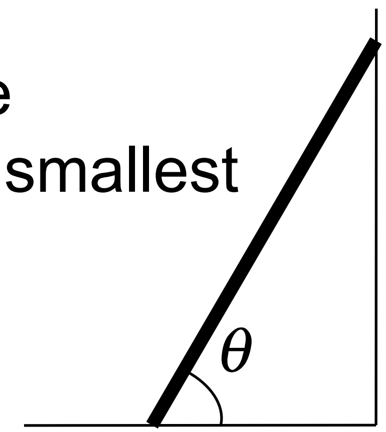

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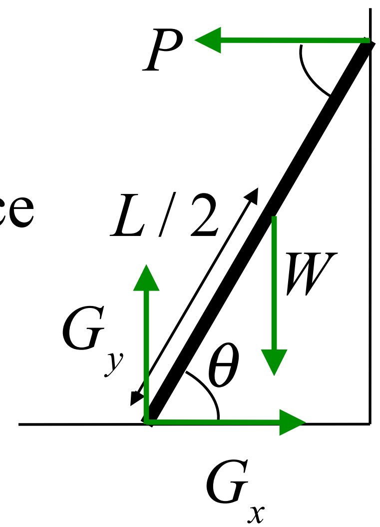

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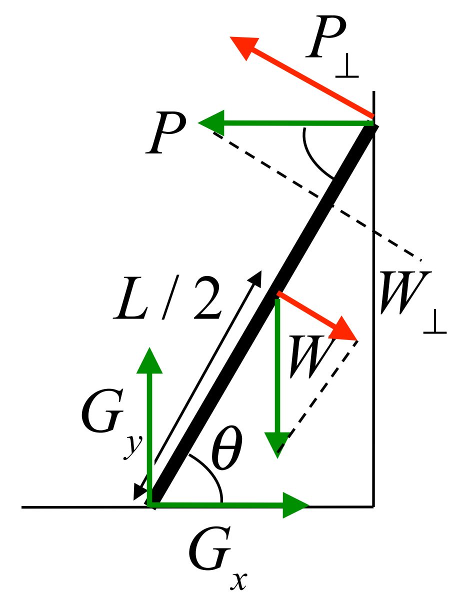
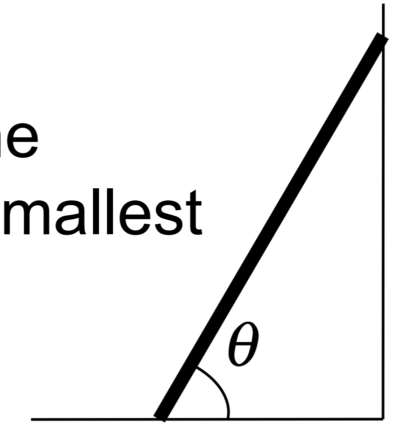


2. Choose pivot point at ground.

3. Find components normal to board

$P_\perp$ and $W_\perp$ are forces producing torque

Example: A board length L lies against a wall. The coefficient of friction with ground 0.650. What is smallest angle the board can be placed without slipping?



4. Net torque must be zero for equilibrium

Torque:

$$\tau_P = +P_\perp L = (P\sin\theta)L$$

$$\tau_W = -W_\perp (L/2) = -(W\cos\theta)(L/2)$$

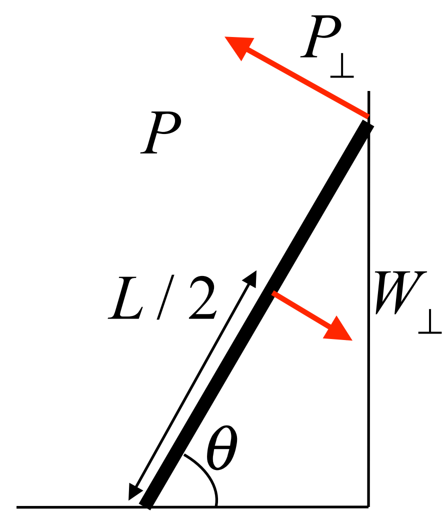
$$\tau_W + \tau_P = 0 \Rightarrow (P\sin\theta) = (W\cos\theta)/2$$

$$W = 2P\sin\theta/\cos\theta$$

Forces:

$$G_y = W$$

$$P = G_x = \mu G_y = \mu W$$



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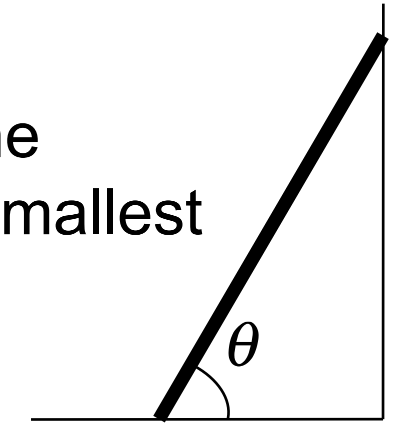

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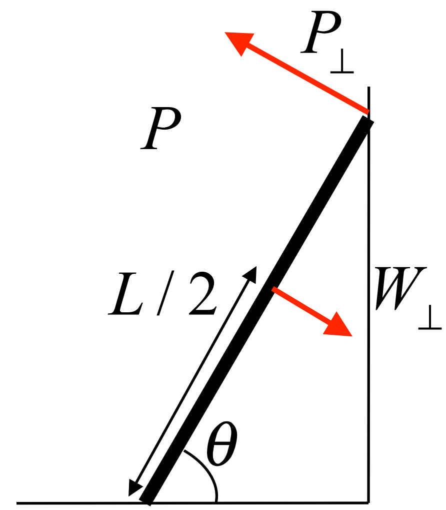


5. Combine torque and force equations

$$W = 2P\sin\theta/\cos\theta = 2\mu W\tan\theta$$
$$\tan\theta = 1/(2\mu) = 1/(1.3) = 0.77$$
$$\theta = 37.6^{\circ}$$

Chapter 8 ***Rotational Kinematics/Dynamics***

Angular motion variables
(with the usual motion equations)

displacement $\theta = s / r \quad \text{(rad.)}$

velocity $\omega = v / r \quad \text{(rad./s)}$

acceleration $\alpha = a / r \quad (\text{rad./s}^2)$

Uniform circular motion

centripetal acceleration $a_C = \dfrac{v^2}{r} = \omega^2 r$

centripetal force $F_C = ma_C = \dfrac{mv^2}{r}$

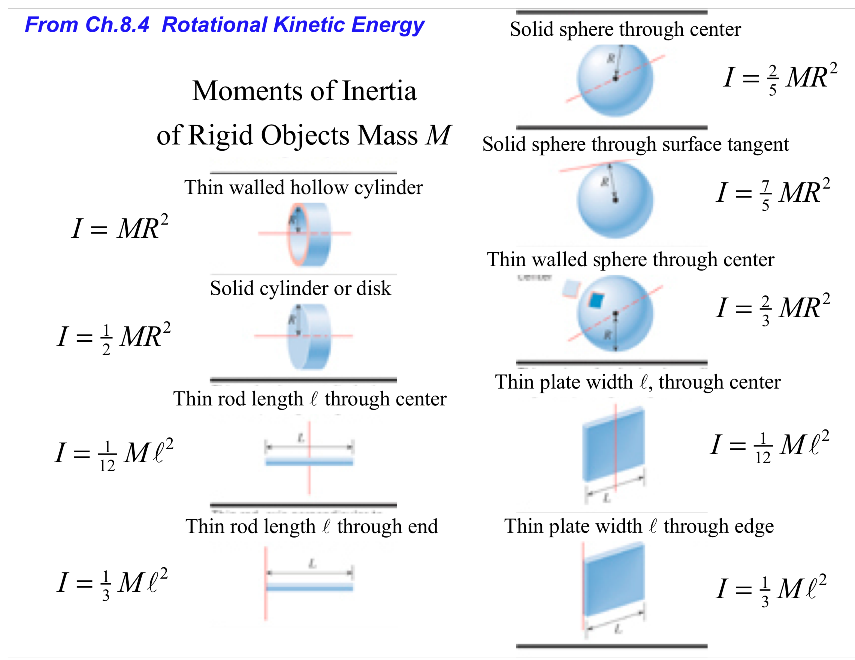
torque (θ : ∢ between $\vec{\mathbf{F}}$ & $\vec{\mathbf{r}}$) $\tau = rF\sin\theta$

Newton's 2nd Law $\tau_{\text{Net}} = I\,\alpha$

rot. kinetic energy $K_{rot} = \frac{1}{2} I\omega^2$

angular momentum $L = I\omega$

I Moment of Inertia



8.8 Newton's Second Law for Rotational Motion About a Fixed Axis

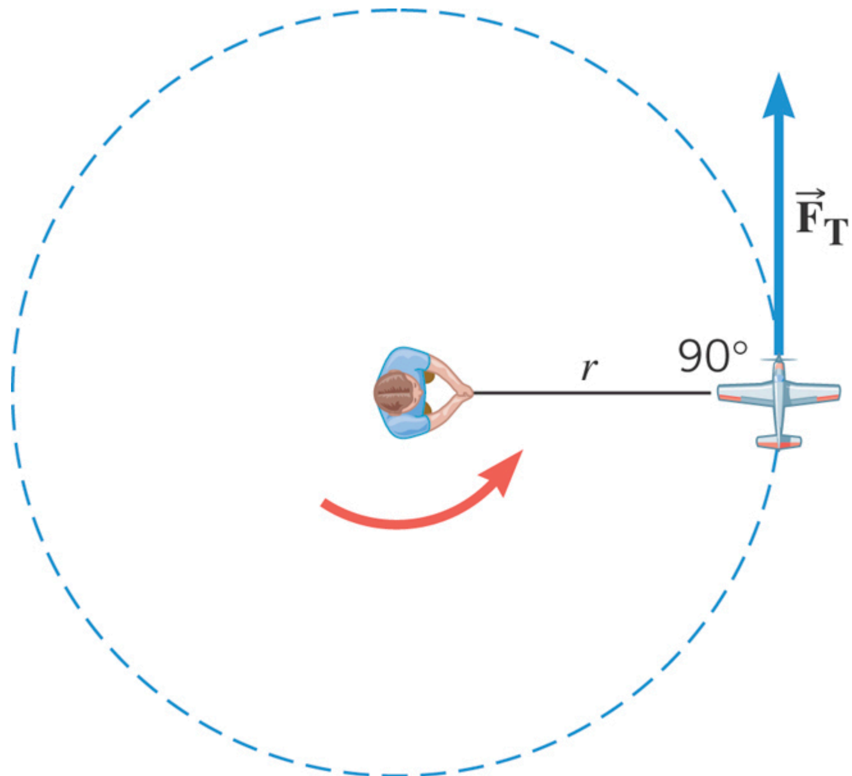
$$\tau = F_T r$$
$$= m a_T r$$
$$= m\alpha r^2$$
$$= \left(mr^2\right)\alpha$$
$$= I\alpha$$

$$F_T = m a_T$$

$$a_T = r\alpha$$

$$I = mr^2$$

Moment of Inertia
of the airplane

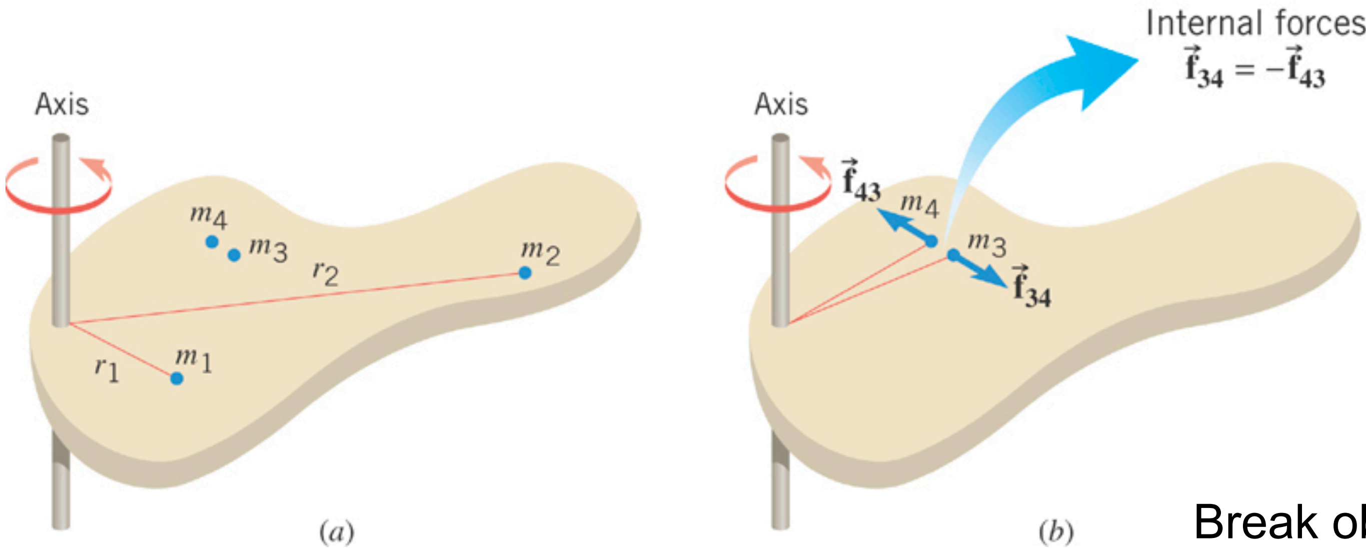


Moment of Inertia, $I = mr^2$, for a point-mass, m, at the end of a massless arm of length, r.

$$\tau = I\alpha$$

Newton's 2nd Law for rotations

8.8 Newton's Second Law for Rotational Motion About a Fixed Axis



Break object into N individual masses

$$\tau_{\text{Net}} = \sum_i (mr^2)_i\, \alpha$$

Net external torque

Moment of inertia

$$\tau_1 = \left(m_1 r_1^2\right)\alpha$$

$$\tau_2 = \left(m_2 r_2^2\right)\alpha$$

$$\vdots$$

$$\tau_N = \left(m_N r_N^2\right)\alpha$$

8.8 Newton's Second Law for Rotational Motion About a Fixed Axis

ROTATIONAL ANALOG OF NEWTON'S SECOND LAW FOR A RIGID BODY ROTATING ABOUT A FIXED AXIS

$$\text{Net external torque} = \begin{pmatrix} \text{Moment of} \\ \text{inertia} \end{pmatrix} \times \begin{pmatrix} \text{Angular} \\ \text{acceleration} \end{pmatrix}$$

$$\tau_{\text{Net}} = I\,\alpha$$

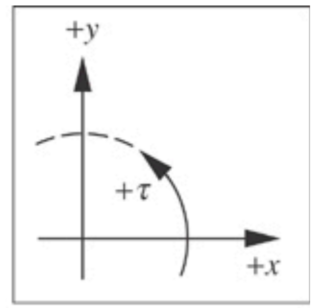
$$I = \sum_i (mr^2)_i$$

Requirement: Angular acceleration must be expressed in radians/s^2.

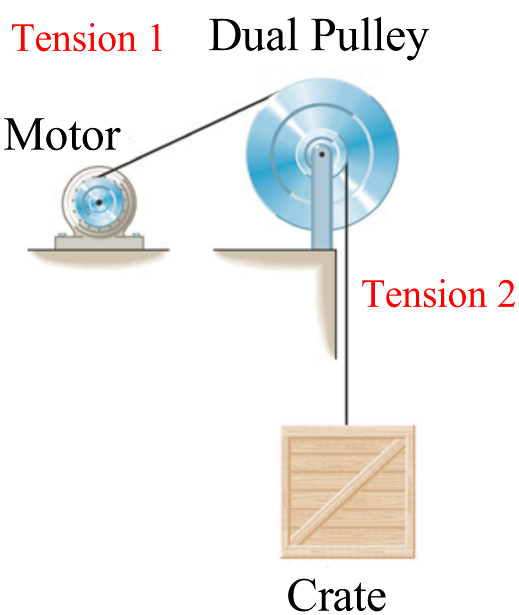
Example: **Hoisting a Crate**

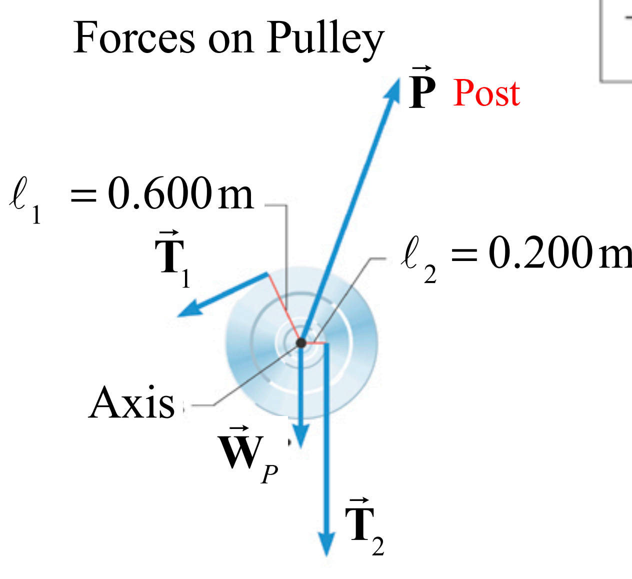
dual pulley

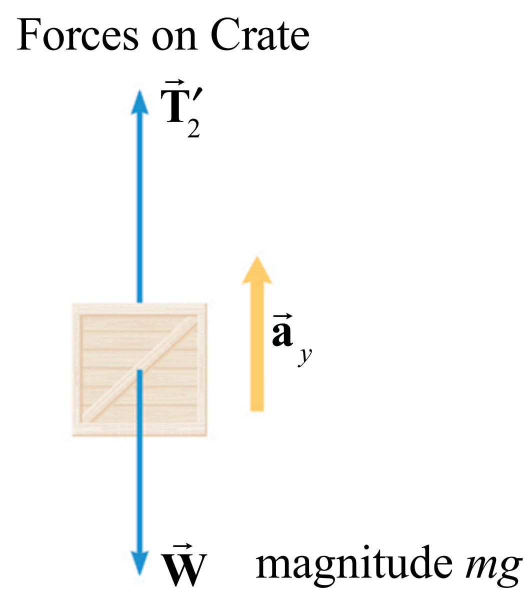
The combined moment of inertia of the dual pulley is 50.0 kg·m^2. The crate weighs 4420 N. A tension of 2150 N is maintained in the cable attached to the motor. Find the angular acceleration of the dual Pulley (radius-1 = 0.600m, radius-2 = 0.200 m).



up is positive







8.8 Newton's Second Law for Rotational Motion About a Fixed Axis

2nd law for linear motion of crate

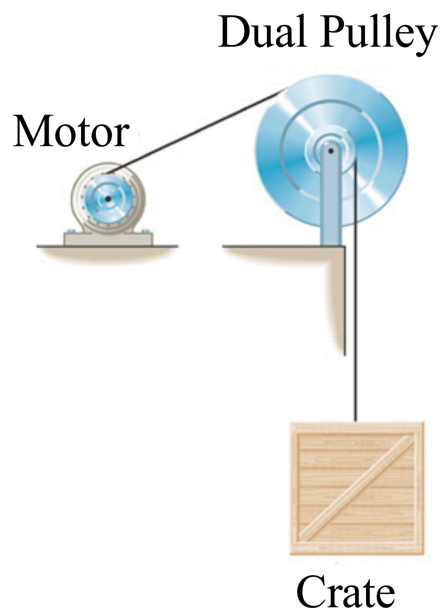
$$\sum F_y = T_2 - mg = ma_y$$

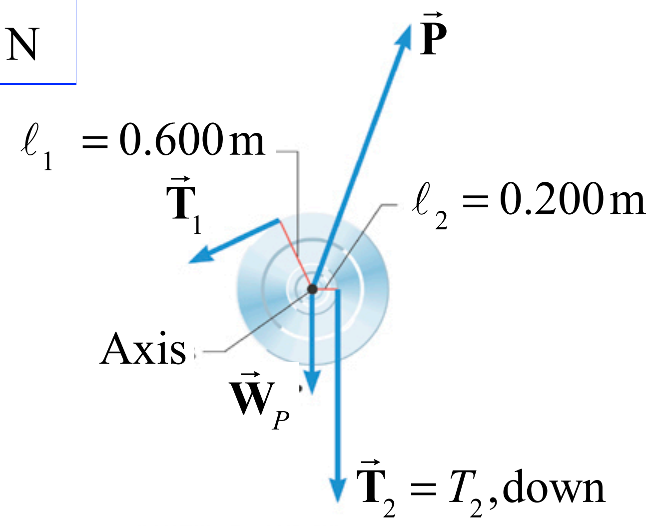
$$a_y = \ell_2 \alpha$$

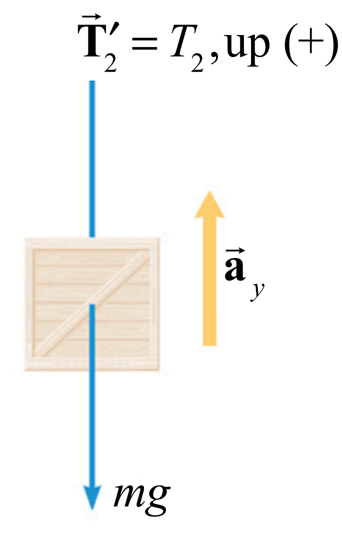
$$I = 46\,\text{kg}\cdot\text{m}^2$$
$$mg = 4420\ \text{N}$$
$$T_1 = 2150\ \text{N}$$

2nd law for rotation of the pulley

$$\sum \tau = T_1\ell_1 - T_2\ell_2 = I\alpha$$







$$T_2 - mg = m\ell_2\alpha$$
$$T_1\ell_1 - T_2\ell_2 = I\alpha$$
$$T_2 = m\ell_2\alpha + mg \ \&\ T_2 = \frac{T_1\ell_1 - I\alpha}{\ell_2}$$
$$m\ell_2^2\alpha + I\alpha = T_1\ell_1 - mg\ell_2$$
$$\alpha = \frac{T_1\ell_1 - mg\ell_2}{m\ell_2^2 + I} = 6.3\ \text{rad/s}^2$$

$$T_2 = m\ell_2\alpha + mg = \left(451(.6)(6.3) + 4420\right)\text{N} = 6125\ \text{N}$$

DEFINITION OF ANGULAR MOMENTUM

The angular momentum *L* of a body rotating about a fixed axis is the product of the body's moment of inertia and its angular velocity with respect to that axis:

$$L = I\omega$$

Requirement: The angular speed must be expressed in rad/s.

SI Unit of Angular Momentum: kg·m^2/s

PRINCIPLE OF CONSERVATION OF ANGULAR MOMENTUM

The angular momentum of a system remains constant (is conserved) if the net external torque acting on the system is zero.

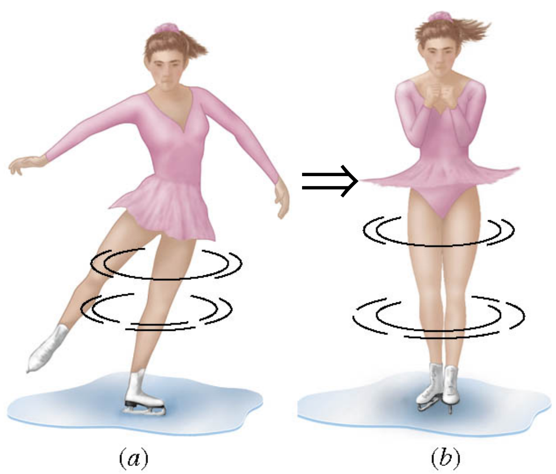
Angular momentum, L

$$L_i = I_i\omega_i; \quad L_f = I_f\omega_f$$

No external torque

$\Rightarrow$ Angular momentum conserved

$$L_f = L_i$$



Moment of Inertia decreases

$$I = \sum mr^2, \; r_f < r_i$$

$$I_f < I_i$$

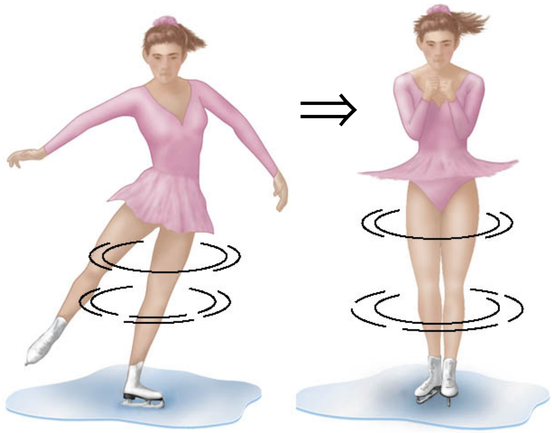
$$\frac{I_i}{I_f} > 1$$

$$I_f\omega_f = I_i\omega_i$$

$$\omega_f = \frac{I_i}{I_f}\omega_i; \quad \frac{I_i}{I_f} > 1$$

$\omega_f > \omega_i$ (angular speed increases)

8.9 Angular Momentum



From Angular Momentum Conservation

$$\omega_f = \left(I_i / I_f\right)\omega_i$$

because $I_i / I_f > 1$

Angular velocity increases

Is Energy conserved?

$$K_f = \tfrac{1}{2} I_f \omega_f^2$$

$$= \tfrac{1}{2} I_f \left(I_i / I_f\right)^2 \omega_i^2$$

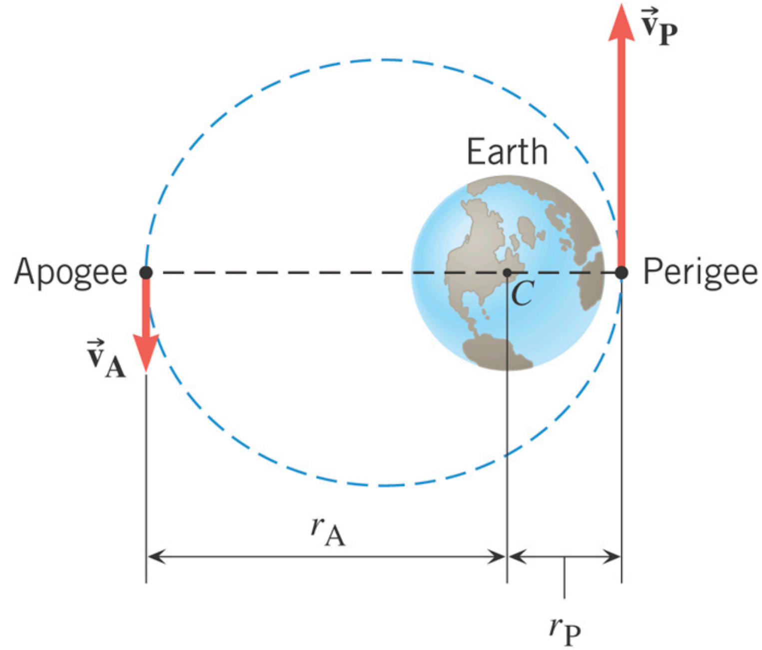
$$= \left(I_i / I_f\right)\left(\tfrac{1}{2} I_i \omega_i^2\right) \qquad K_i = \tfrac{1}{2} I_i \omega_i^2;$$

$$= \left(I_i / I_f\right) K_i \quad \Rightarrow \quad \text{Kinetic Energy increases}$$

Energy is NOT conserved because pulling in the arms does (NC) work on the mass of each arm and increases the kinetic energy of rotation.

8.9 Angular Momentum

Example: A Satellite in an Elliptical Orbit



An artificial satellite is placed in an elliptical orbit about the earth. Its point of closest approach is 8.37x10^6 m from the center of the earth, and its point of greatest distance is 25.1x10^6 m from the center of the earth.The speed of the satellite at the perigee is 8450 m/s. Find the speed at the apogee.

$$I_A = mr_A^2; \quad I_P = mr_P^2$$

$$\omega_A = v_A / r_A; \quad \omega_P = v_P / r_P$$

Gravitational force along r (no torque) $\Rightarrow$ Angular momentum conserved

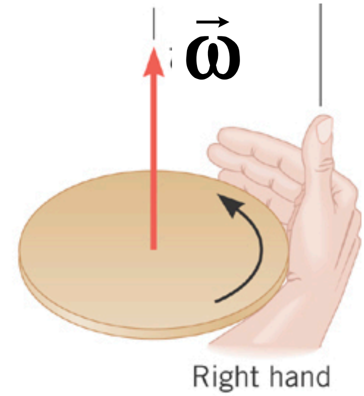
$$I_A \omega_A = I_P \omega_P$$

$$mr_A^2 \left(v_A / r_A\right) = mr_P^2 \left(v_P / r_P\right) \quad \Rightarrow \quad r_A v_A = r_P v_P$$

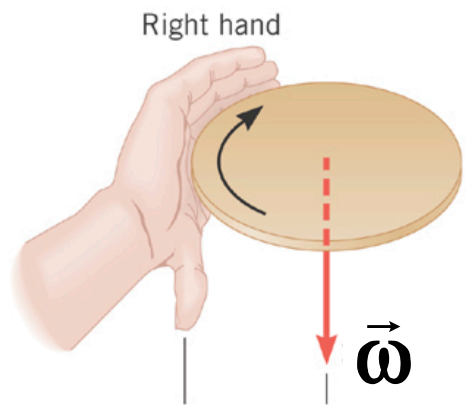
$$v_A = \left(r_P / r_A\right) v_P = \left[(8.37 \times 10^6) / (25.1 \times 10^6)\right] \left(8450 \text{ m/s}\right) = 2820 \text{ m/s}$$

8.9 *The Vector Nature of Angular Variables*

Right-Hand Rule: Grasp the axis of rotation with your right hand, so that your fingers circle the axis in the same sense as the rotation.



Your extended thumb points along the axis in the direction of the angular velocity.



<u>Vector Quantities in Rotational Motion</u>

Angular Acceleration $$\vec{\alpha} = \frac{\Delta\vec{\omega}}{\Delta t}$$

Angular Momentum $$\vec{L} = I\vec{\omega}$$

Torque $$\vec{\tau} = I\,\vec{\alpha} = \frac{\Delta\vec{L}}{\Delta t}$$ Changes Angular Momentum

If torque is perpendicular to the angular momentum, only the direction of the angular momentum changes (precession) – no changes to the magnitude.

Rotational/Linear Dynamics Summary

linear		rotational
x	displacement	θ
v	velocity	ω
a	acceleration	α
m	point m inertia	$I = mr^2$
F	force/torque	$\tau = Fr\sin\theta$

Uniform circular motion

$$a_C = v_T^2/r \qquad F_C = mv_T^2/r$$

linear	rotational
$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$	$\vec{\tau} = I\vec{\alpha}$
$W = F(\cos\theta)x$	$W_{rot} = \tau\phi$
$K = \frac{1}{2}mv^2$	$K_{rot} = \frac{1}{2}I\omega^2$
$W \Rightarrow \Delta \mathrm{K}$	$W_{rot} \Rightarrow \Delta \mathrm{K}_{rot}$
$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	$\vec{L} = I\vec{\omega}$
$\vec{\mathbf{F}}\Delta t = \Delta\vec{\mathbf{p}}$	$\vec{\tau}\Delta t = \Delta\vec{L}$

Potential Energies

$U_G = mgy$

or $U_G = -GmM_E/r$

$U_S = \frac{1}{2}kx^2$

Conservation laws

Conserved:

If $W_{NC} = 0$, $E = K + U$

If $\mathbf{F}_{ext} = 0$, $\mathbf{P}_{system} = \sum \mathbf{p}$

If $\tau_{ext} = 0$, $\vec{L} = I\vec{\omega}$