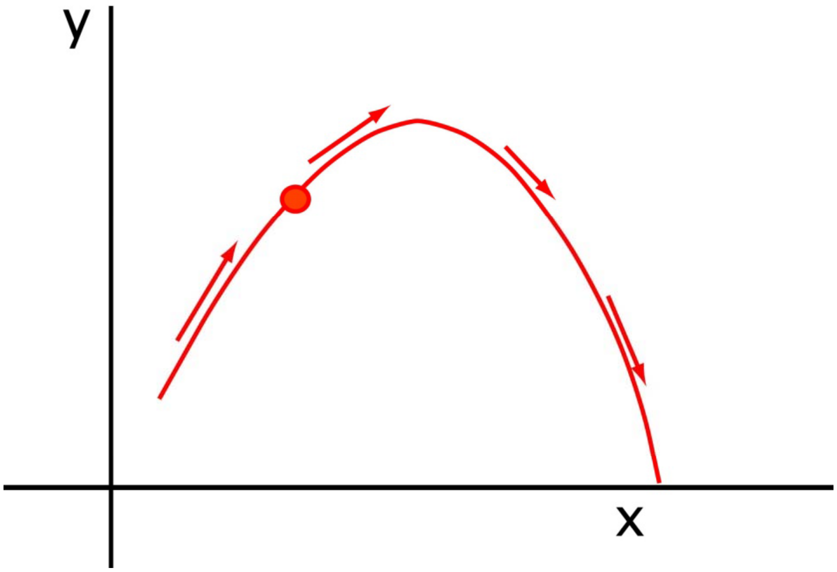
Classical Dynamics for a System of Particles
(Chapter 9)
Momentum and the Center of Mass



- **Toss a small pebble**.
It will follow a parabolic trajectory, as shown.

The momentum is, by definition, $\mathbf{p} = m\,\mathbf{v}$.

The x component is constant,

$$p_x = m\,v_0 \cos\theta\ ;$$

the y component decreases at the rate $-\,mg$,

$$p_y = m\,v_0 \sin\theta \;-\; m\,g\,t\ .$$

The momentum vector is

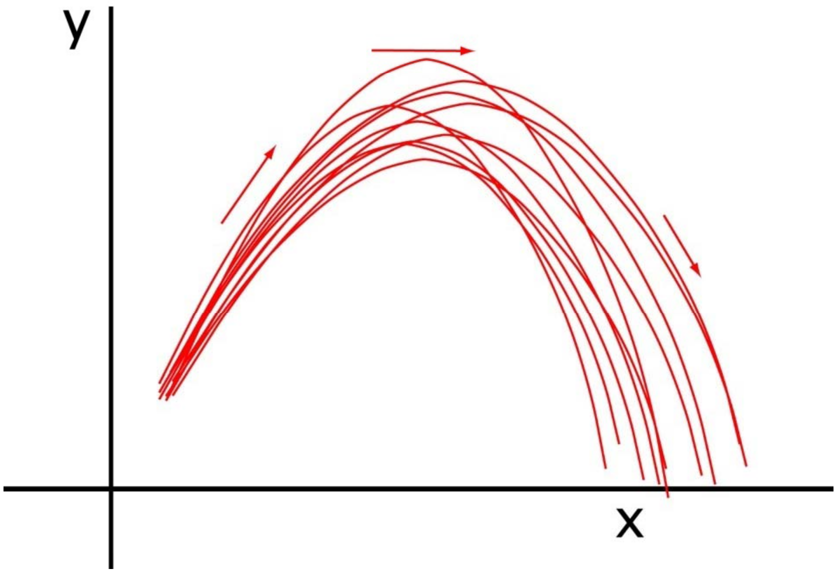
$$\mathbf{p}(t) = p_{0x}\,\mathbf{i} + (\,p_{0y} \;-\; m\,g\,t\,)\,\mathbf{j}\ ; \qquad \textbf{(1)}$$

it is consistent with Newton's second law,

$$d\mathbf{p}\,/\,dt = -\,m\,g\,\mathbf{j} \quad (\text{the } \textit{weight}) \qquad \textbf{(2)}$$

All this is a familiar example of single-particle dynamics.

- **Now toss a handful of small pebbles.**



Each pebble will probably move independently. (Two pebbles could hit each other as they move, but that would be unlikely; we'll ignore that possibility.) Then we can write the momentum of the i-th pebble as

$$\mathbf{p_i}(t) = p_{0ix}\,\mathbf{i} + (\,p_{0iy} \;-\; m_i\,g\,t\,)\,\mathbf{j}\;,$$

consistent with Newton's second law,

$$d\mathbf{p_i}\,/\,dt = -\,m_i\,g\,\mathbf{j} \quad \text{(the \textit{weight of pebble \#i})}$$

We know the complete dynamics of the system because the motions of the pebbles are independent

The **total momentum** is defined by

$$\mathbf{P}(t) = \sum_{i=1}^{N} \mathbf{p}_i(t) \quad ;$$

by the complete dynamics,

$$\mathbf{P}(t) = P_{0x}\, \boldsymbol{i} + (P_{0y} - Mg)\boldsymbol{j} \qquad \textbf{(3)}$$

where $\mathbf{P}_0$ is the initial momentum and M is the total mass, $M = \Sigma\, m_i$. Compare equations (**3**) and (**1**). You should see that the ***total momentum*** of the N pebbles is mathematically the same as if there were only one large pebble with mass M.

The **center of mass** is the "average position of the system of particles, weighted by their masses."
That is,

$$\mathbf{X}(t) = \frac{\sum_{i=1}^{N} m_i \mathbf{x}_i(t)}{\sum_{i=1}^{N} m_i}$$

or, $$\mathbf{X}(t) = \frac{1}{M}\left[m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3 + \cdots m_N\mathbf{x}_N\right]$$

An important theorem relates the total momentum and center of mass:

$$\mathbf{P} = \mathrm{M}\,\mathbf{V} \qquad \text{where} \qquad \mathbf{V} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{dt}} \qquad (4)$$

Here's the proof …

The center of mass position is

$$\vec{X} = \frac{1}{M}\left\{ m_1\vec{x}_1 + m_2\vec{x}_2 + m_3\vec{x}_3 + \dots + m_N\vec{x}_N \right\}$$

The center of mass velocity is:

$$\vec{V} = \frac{d\vec{X}}{dt} = \frac{1}{M}\left\{ m_1\frac{d\vec{x}_1}{dt} + m_2\frac{d\vec{x}_2}{dt} + \dots + m_N\frac{d\vec{x}_N}{dt} \right\}$$

$$= \frac{1}{M}\left\{ \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots + \vec{p}_N \right\}$$

$$= \frac{1}{M}\,\vec{P} \qquad \text{(total momentum).}$$

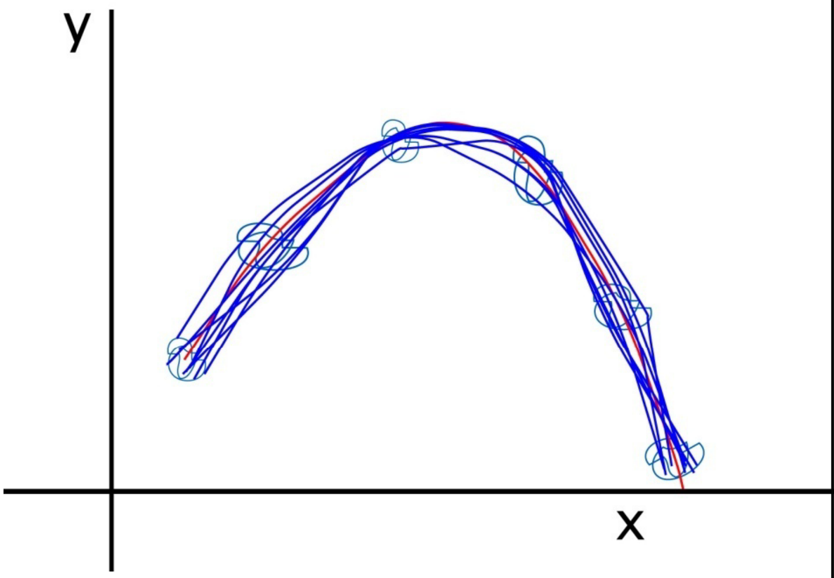
Thus

$$\vec{P} = M\vec{V} \qquad \text{where} \qquad \vec{V} = \frac{d\vec{X}}{dt}$$

Q.E.D.

• Now toss a handful of strongly-interacting pebbles.

Suppose that some pebbles are tied together by strings. Suppose that some pebbles are connected by springs, initially stretched or compressed. You are not now tossing a set of N independent particles, but a ***big blob*** of pebbles that are exerting forces on each other.



But we can determine the total momentum **P(t)** and the center of mass trajectory **X(t).** The remarkable result is that the total momentum and the center of mass trajectory are exactly the same as if the pebbles were not exerting forces on each other:

$$\mathbf{P}(t) = P_{0x}\hat{\mathbf{i}} + (P_{0y} - Mgt)\hat{\mathbf{j}} \qquad (5)$$

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{P}(t)}{M} \qquad (6)$$

Note that equations (5) and (6) (for strongly interacting particles) are the same as equations (3) and (4) (for non-interacting particles). *Why are the total momentum and center of mass independent of the internal forces in the system of particles?*

Proof of equation (5)

Total momentum $\vec{P} = \sum_{i=1}^{N} \vec{p}_i$

Consider the derivative,

$$\frac{d\vec{P}}{dt} = \sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{N} \vec{F}_i$$

← Newton's second law

- $\vec{F}_i = \vec{F}_i^{ext} + \sum_{j=1}^{N} \vec{f}_{ij}$

where $\vec{F}_i^{ext} = -m_i g \hat{j}$

and $\vec{f}_{ij}$ = the force on mass i due to the interaction of masses i and j

So

$$\sum_{i=1}^{N} \vec{F}_i = \underbrace{\sum_{i=1}^{N} \vec{F}_i^{ext}}_{= -Mg\hat{j} \ (M = \text{total mass})} + \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} \vec{f}_{ij}}_{= 0 \text{ by Newton's third law}}$$

⇑

That's the crucial idea for analyzing the dynamics of a system of particles.

Result

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext} = -Mg\hat{j}$$

$$\therefore \vec{P}(t) = P_{0x}\hat{i} + (P_{0y} - Mgt)\hat{j}$$

Q.E.D.

Proof of equation (6)

The center of mass position is

$$\vec{X} = \frac{1}{M}\left\{ m_1 \vec{x}_1 + m_2 \vec{x}_2 + m_3 \vec{x}_3 + \ldots + m_N \vec{x}_N \right\}$$

The center of mass velocity is

$$\vec{V} = \frac{d\vec{X}}{dt} = \frac{1}{M}\left\{ m_1 \frac{d\vec{x}_1}{dt} + m_2 \frac{d\vec{x}_2}{dt} + \cdots + m_N \frac{d\vec{x}_N}{dt} \right\}$$

$$= \frac{1}{M}\left\{ \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N \right\}$$

$$= \frac{1}{M} \vec{P} \quad \text{(total momentum).}$$

Thus

$$\vec{P} = M\vec{V} \quad \text{where} \quad \vec{V} = \frac{d\vec{X}}{dt}$$

Q.E.D.

These examples — the non-interacting pebbles and the blob of strongly interacting pebbles — illustrate something important about the dynamics of a system of particles. Questions about the ***total momentum*** and ***center of mass*** may have simple answers, independent of complicated internal dynamics.

Exercise: Prove generally, i.e., for ***any*** system of particles …

$$\mathbf{P} = M\mathbf{V}_{\mathrm{CM}}$$

and

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{\mathrm{ext}}$$

… where M = the total mass, **P** = the total momentum, **V** = d**X**/dt = velocity of the center of mass point, and $\mathbf{F}^{\mathrm{ext}}$ = the sum of all *external* forces.

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Reading and problem assignments:
❖  Read Chapter 9 from the
textbook, Thornton and Marion,
Classical Dynamics.
❖  Do the LON-CAPA problems
entitled “Homework Set 3a”.
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Classical Dynamics for a System of Particles (Chapter 9)

The two-body problem

Here's an interesting fact ---
The dynamics of an isolated system of two particles is equivalent to the dynamics of a single particle.

So we can always solve the 2-body problem — it reduces to a 1-body problem.

The equations of motion are

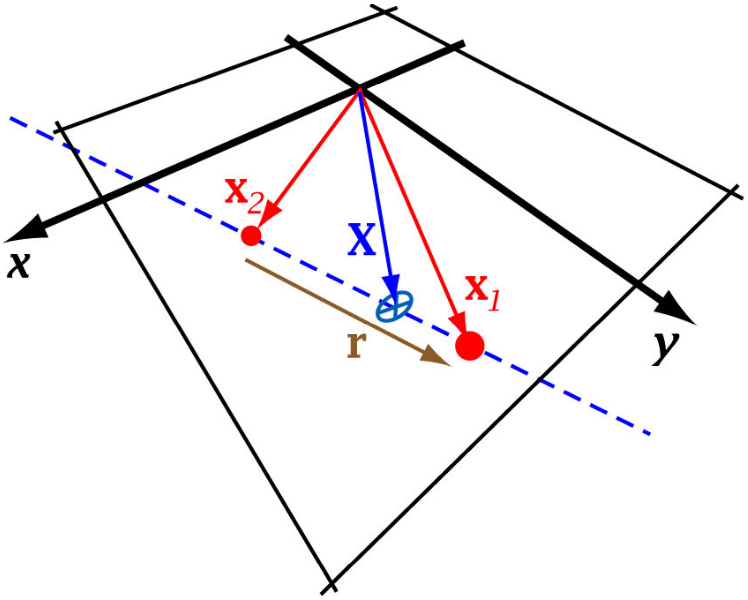
$$m_1 \frac{d\vec{v}_1}{dt} = \vec{F}_1 \qquad \text{and} \qquad \vec{F}_1 = \vec{F}_1^{ext} + \vec{f}$$

$$m_2 \frac{d\vec{v}_2}{dt} = \vec{F}_2 \qquad \qquad \vec{F}_2 = \vec{F}_2^{ext} - \vec{f}$$

For an isolated system, the external forces would be 0,

$$\vec{F}_1^{ext} = 0 \quad \text{and} \quad \vec{F}_2^{ext} = 0. \qquad \text{(isolated system)}$$

In the case of an isolated system, we can reduce the two differential equations to a single differential equation (= the equivalent 1-body problem).



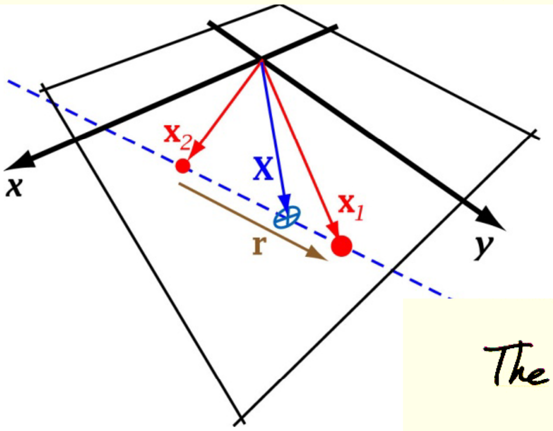
Masses m_1 and m_2 have position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. The center of mass position is $\mathbf{X}$; the relative vector is $\mathbf{r}$.

Separation of CM motion and relative motion

$$m_1 \frac{d\vec{v}_1}{dt} = \vec{f} \quad \text{where} \quad \vec{v}_1 = \frac{d\vec{x}_1}{dt} \qquad (1)$$

$$m_2 \frac{d\vec{v}_2}{dt} = -\vec{f} \quad \text{where} \quad \vec{v}_2 = \frac{d\vec{x}_1}{dt} \qquad (2)$$

We'll now separate the center of mass (CM) motion and relative motion.



Masses m_1 and m_2 have position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. The center of mass position is $\mathbf{X}$; the relative vector is $\mathbf{r}$.

The center of mass and relative vectors are defined in the FIGURE; that is,

$$\vec{X} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \qquad (3)$$

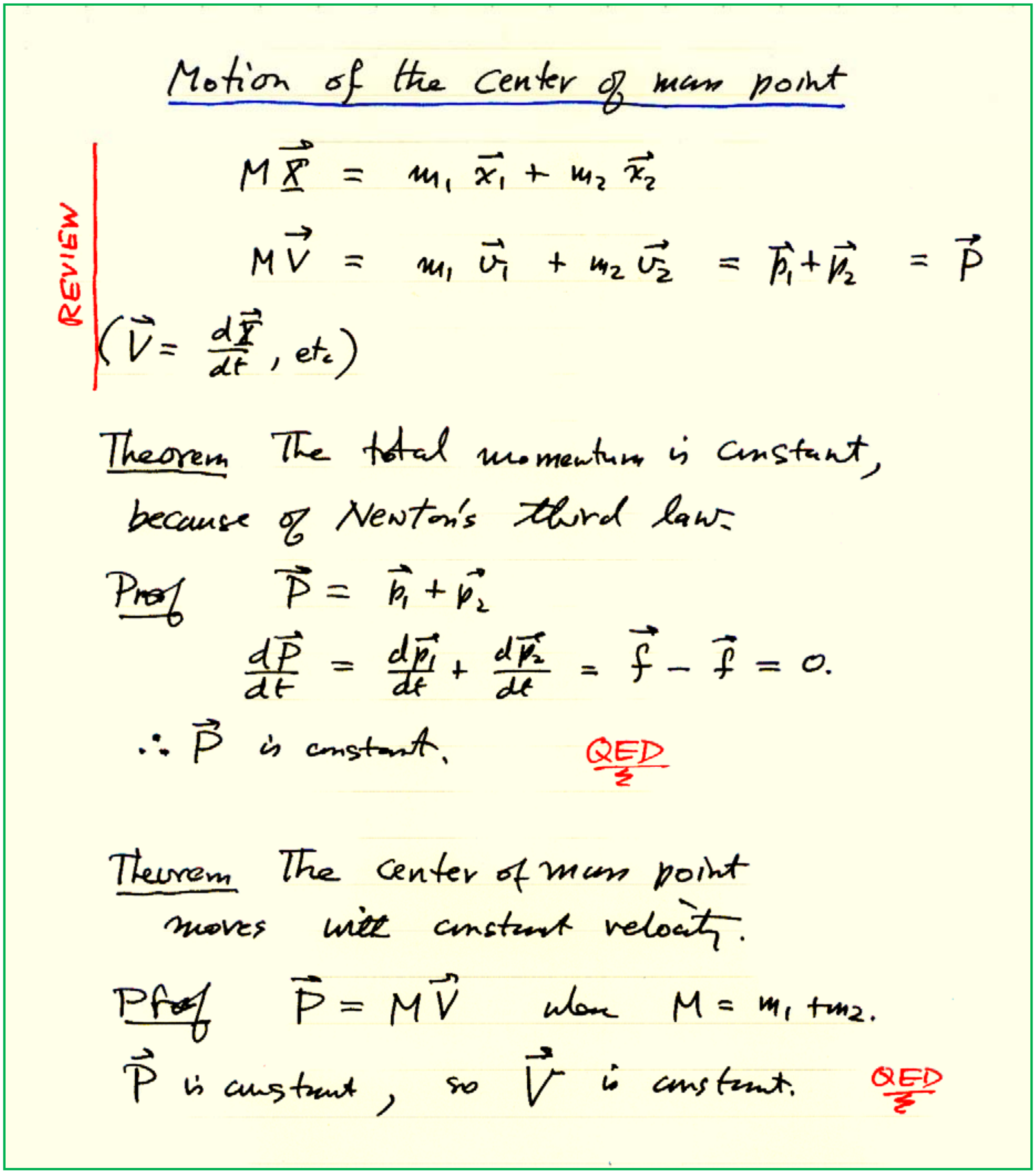
$$\vec{r} = \vec{x}_1 - \vec{x}_2 \qquad (4)$$

Or, we may express the particle position vectors in terms of $\vec{X}$ and $\vec{r}$

$$\vec{x}_1 = \vec{X} + \frac{m_2}{M} \vec{r} \qquad (5)$$

$$\vec{x}_2 = \vec{X} - \frac{m_1}{M} \vec{r} \, . \qquad (6)$$

Please verify that equations (3) and (4) follow from equations (5) and (6).



Motion of the center of mass point

REVIEW

$$M\vec{\underline{X}} = m_1 \vec{x}_1 + m_2 \vec{x}_2$$

$$M\vec{V} = m_1 \vec{v}_1 + m_2 \vec{v}_2 = \vec{p}_1 + \vec{p}_2 = \vec{P}$$

$\left(\vec{V} = \frac{d\vec{X}}{dt}, \text{etc}\right)$

Theorem The total momentum is constant, because of Newton's third law.

Proof $\vec{P} = \vec{p}_1 + \vec{p}_2$

$$\frac{d\vec{P}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \vec{f} - \vec{f} = 0.$$

$\therefore \vec{P}$ is constant. QED

Theorem The center of mass point moves with constant velocity.

Proof $\vec{P} = M\vec{V}$ where $M = m_1 + m_2$.

$\vec{P}$ is constant, so $\vec{V}$ is constant. QED

So the motion of the center of mass is <u>constant velocity</u>: **X**(t) = **V** t and **V** = **P**/M .

The relative motion

RELATIVE VECTOR

$$\vec{r} = \vec{x}_1 - \vec{x}_2$$

Define $\vec{v} = \frac{d\vec{r}}{dt}$

Theorem $\mu \frac{d\vec{v}}{dt} = \vec{f}$

where μ is the reduced mass.

Proof $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{x}_1}{dt} - \frac{d\vec{x}_2}{dt} = \vec{v}_1 - \vec{v}_2$

$$\frac{d\vec{v}}{dt} = \frac{d\vec{v}_1}{dt} - \frac{d\vec{v}_2}{dt} = \frac{\vec{f}}{m_1} - \frac{-\vec{f}}{m_2} = \frac{\vec{f}}{\mu}$$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ or $\mu = \frac{m_1 m_2}{m_1 + m_2}$

QED

Reduction of the isolated 2-body problem to an equivalent 1-body problem

$\vec{x}_1 = \vec{X} + \frac{m_2}{M}\vec{r}$ and $\vec{x}_2 = \vec{X} - \frac{m_1}{M}\vec{r}$

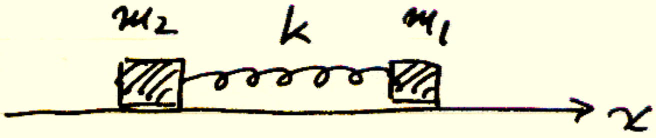
Here $\vec{X}(t) = \vec{X}_0 + \vec{V}_0 t$

and $\mu \frac{d^2\vec{r}}{dt^2} = \vec{f}$

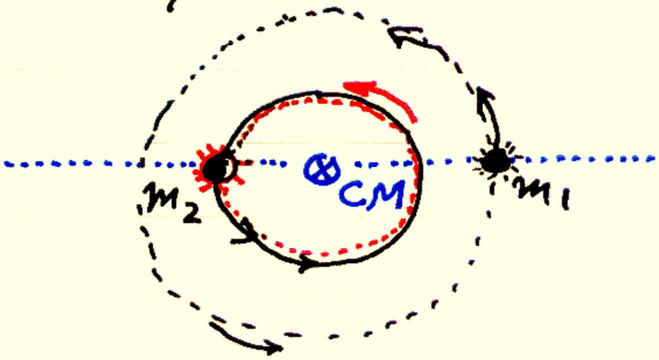
Solve $\mu \ddot{\vec{r}} = \vec{f}$; then $\vec{x}_1$ & $\vec{x}_2$ are known.

The lecture will continue with 2 examples.

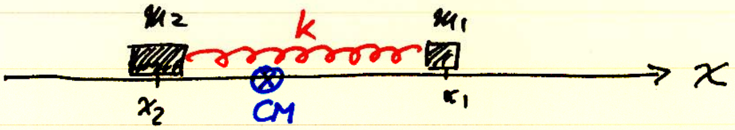
Example 1 Diatomic Oscillator



Example 2 Binary Star System



Example 1. Diatomic Oscillator



Equations of motion

ℓ : equilibrium length

$$m_1 \ddot{x}_1 = F = -k(x_1 - x_2 - \ell)$$

$$m_2 \ddot{x}_2 = -F = k(x_1 - x_2 - \ell)$$

The equivalent 1-body problem ($r = x_1 - x_2$)

r: distance between the masses

$$\mu \ddot{r} = F = -k(r - \ell)$$

Let $r = \ell + \epsilon(t)$; then $\mu \ddot{\epsilon} = -k\epsilon$ ← harmonic oscillator

$\epsilon(t) = A \cos\omega t$ where $\omega^2 = \frac{k}{\mu}$

Result The frequency is $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$

$$f = \frac{1}{2\pi}\sqrt{\frac{k(m_1+m_2)}{m_1 \cdot m_2}}$$

Examples • If $m_1 = m_2 = m$ then $f = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

• If $m_2 \gg m_1$ then $f \approx \frac{1}{2\pi}\sqrt{\frac{k}{m_1}}$

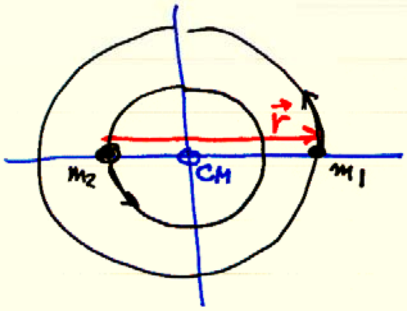
Comment Consider the frame of reference where CM ⊛ is at rest at $x = 0$.

$x_1 = X + \frac{m_2}{M} r = \frac{m_2}{M} r$ so $x_1 = \frac{m_2}{M}(\ell + \epsilon)$

$$\therefore m_1 \ddot{x}_1 = \frac{m_1 m_2}{M}\left(\frac{-k}{\mu}\epsilon\right) = -k(r - \ell)$$

$$= -k(x_1 - x_2 - \ell) \text{ as required.}$$

Example : Binary Star System with Circular Orbits



Use the frame of reference where the CM point is at rest at (0,0).

$$\vec{x}_1 = \vec{X} + \frac{m_2}{M}\vec{r} = \frac{m_2}{M}\vec{r}$$

$$\vec{x}_2 = \vec{X} - \frac{m_1}{M}\vec{r} = -\frac{m_1}{M}\vec{r}$$

The stars revolve around the center of mass point on circular orbits with radii $R_1 = \frac{m_2}{M} r$ and $R_2 = \frac{m_1}{M} r$; and r is the distance between the stars, $= R_1 + R_2$.

The equivalent 1-body problem

$$\mu \ddot{\vec{r}} = \vec{f} \quad \Rightarrow \quad \mu \ddot{\vec{r}} = -\frac{G m_1 m_2}{r^2}\,\hat{r}$$

The imaginary particle (μ) undergoes uniform circular motion with radius r, so

$$\mu \frac{v^2}{r} = \frac{G m_1 m_2}{r^2}$$

period

$$\therefore \quad v = \sqrt{\frac{G(m_1+m_2)}{r}} \quad \text{and} \quad T = \frac{2\pi r}{v} = \sqrt{\frac{4\pi^2 r^3}{G(m_1+m_2)}}$$

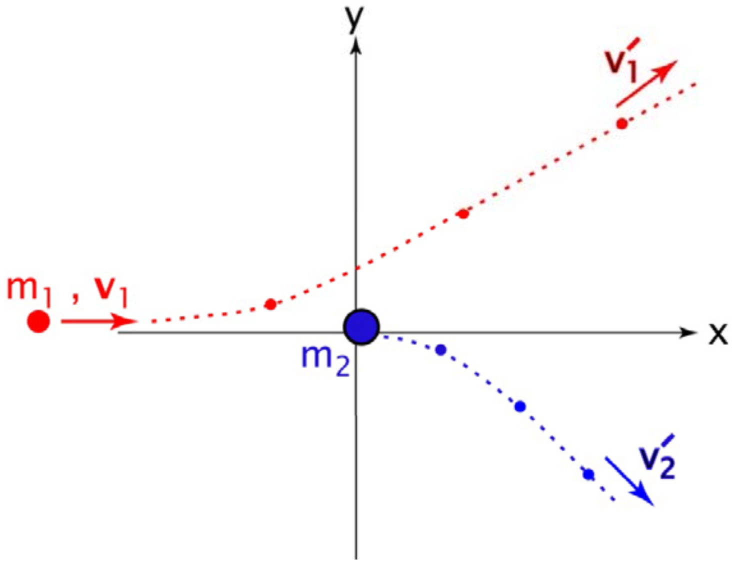
3c - Collisions

In everyday life, we normally think of a collision as an event in which two objects hit each other. In physics the word is used in a more general way. A collision is an event in which:

☐ Two objects move together, experience equal but opposite forces, and accelerate in response to those forces.

☐ When the two objects are far apart, they move freely, i.e., with constant velocity.

1



✓ **Total momentum** is conserved. You should be able to prove that … $m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = m_1 \mathbf{v}_1' + m_2 \mathbf{v}_2'$, because of Newton's third law.

✓ The **center of mass point** moves with constant velocity,

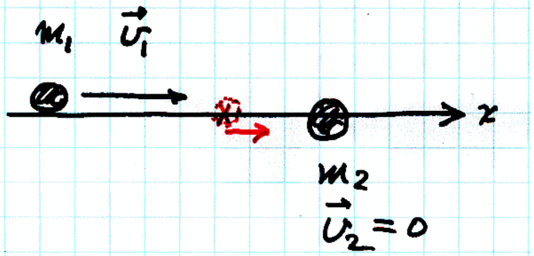
$$\mathbf{V} = \mathbf{P}/M = \text{constant} ;$$

again, you should be able to prove that $\mathbf{V} = \mathbf{P}/M$.

2

The LAB frame and the CENTER of MASS frame

The LAB frame is the frame of reference in which the first particle is the projectile and the second particle is the target, which is initially at rest.



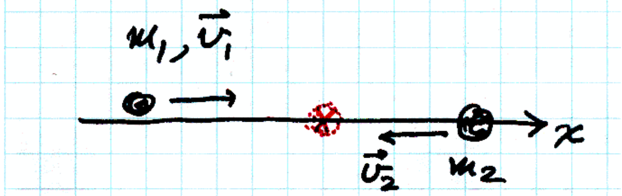
$$\vec{v}_{2L} = 0$$

$$\vec{P}_L = m_1 \vec{v}_{1L}$$

$$\vec{V}_L = \frac{m_1 \vec{v}_{1L}}{m_1 + m_2}$$

1

The CENTER of MASS frame is the frame of reference in which the center of mass of the two particles is at rest.



$$m_1 \vec{v}_{1c} + m_2 \vec{v}_{2c} = 0$$

$$\vec{P}_c = 0$$

$$\vec{V}_c = 0$$

2

LAB ⇔ CM transformations

$$\mathbf{v}_{1c} = \mathbf{v}_{1L} - \mathbf{V}_L = \frac{m_2}{m_1 + m_2}\mathbf{v}_{1L}$$

$$\mathbf{v}_{2c} = \mathbf{v}_{2L} - \mathbf{V}_L = \frac{-m_1}{m_1 + m_2}\mathbf{v}_{1L}$$

3

Totally Inelastic Collisions

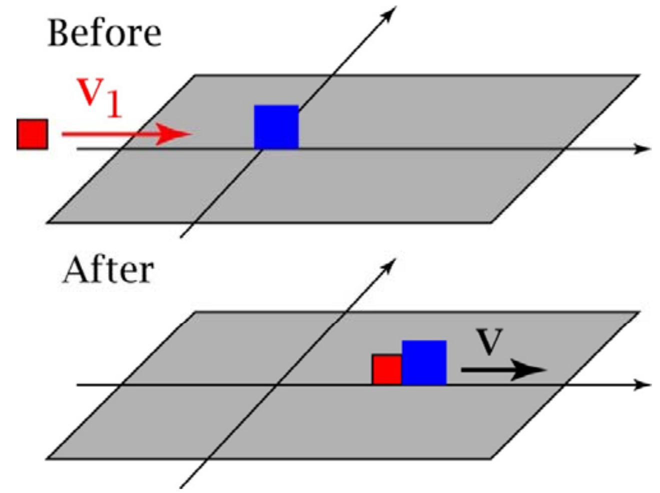
A <u>totally inelastic collision</u> is a collision in which the two particles stick together after the collision.

☐ Total momentum is conserved.

☐ Total kinetic energy is not conserved.

1

Example. Consider a totally inelastic collision with the target initially at rest.



N. B. The scattering angle is necessarily 0 in a totally inelastic collision.

2

Momentum is conserved, so $m_1 v_1 = (m_1 + m_2) v'$

$$v' = \frac{m_1 v_1}{m_1 + m_2}$$

Exercise: Show that the change of kinetic energy is …

$$\frac{\Delta K}{K_{\text{in}}} = \frac{-m_2}{m_1 + m_2}$$

3

Totally Elastic Collisions

A <u>totally elastic collision</u> is a collision in which the total kinetic energy is conserved.

☐ Total momentum is conserved.

☐ Total kinetic energy is conserved.

1

<u>Example.</u> Consider a totally elastic collision in one dimension, with the target initially at rest.

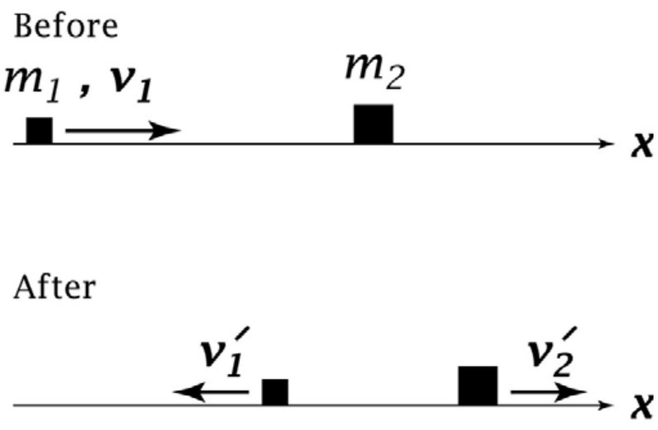
$$m_1 v_1 = m_1 \boldsymbol{v_1}' + m_2 \boldsymbol{v_2}'$$

$$\tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 \boldsymbol{v_1}'^{\boldsymbol{2}} + \tfrac{1}{2} m_2 \boldsymbol{v_2}'^{\boldsymbol{2}}$$

Algebra: Please solve the equations for the final velocities, v_1’ and v_2’ .

Results:

2



$$\boldsymbol{v_1}' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \quad \text{and} \quad \boldsymbol{v_2}' = \frac{2m_1}{m_1 + m_2} v_1$$

<u>Exercises:</u>

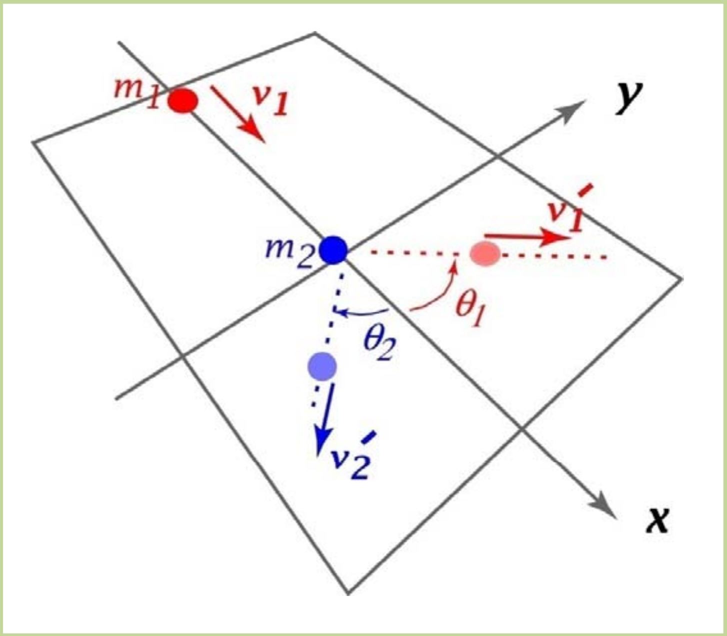
(1) Describe the final state if (a) $m_1 < m_2$; (b) $m_1 = m_2$; (c) $m_1 > m_2$.

(2) What happens if $m_1 << m_2$; what if $m_1 >> m_2$?

(3) Suppose a car is traveling at 30 mph; you toss a ping pong ball in front of the car; and the windshield hits the ping pong ball. How fast will the ping pong ball be moving after the collision?

3

Example. A totally elastic collision in two dimensions (lab frame)



Conservation laws:

$$P_x = m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2$$

$$P_y = 0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2$$

$$K = \tfrac{1}{2} m_1 v_1^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2$$

E.g., given v_1 and θ_1 … calculate v_1', v_2', and θ_2.

Example: Suppose $m_1 = m_2$ in a totally elastic collision. Show that the angle between the final velocities is 90 degrees.

Solution: The masses are equal so the conservation of momentum is … $\boldsymbol{v_1 = v_1' + v_2'}$.

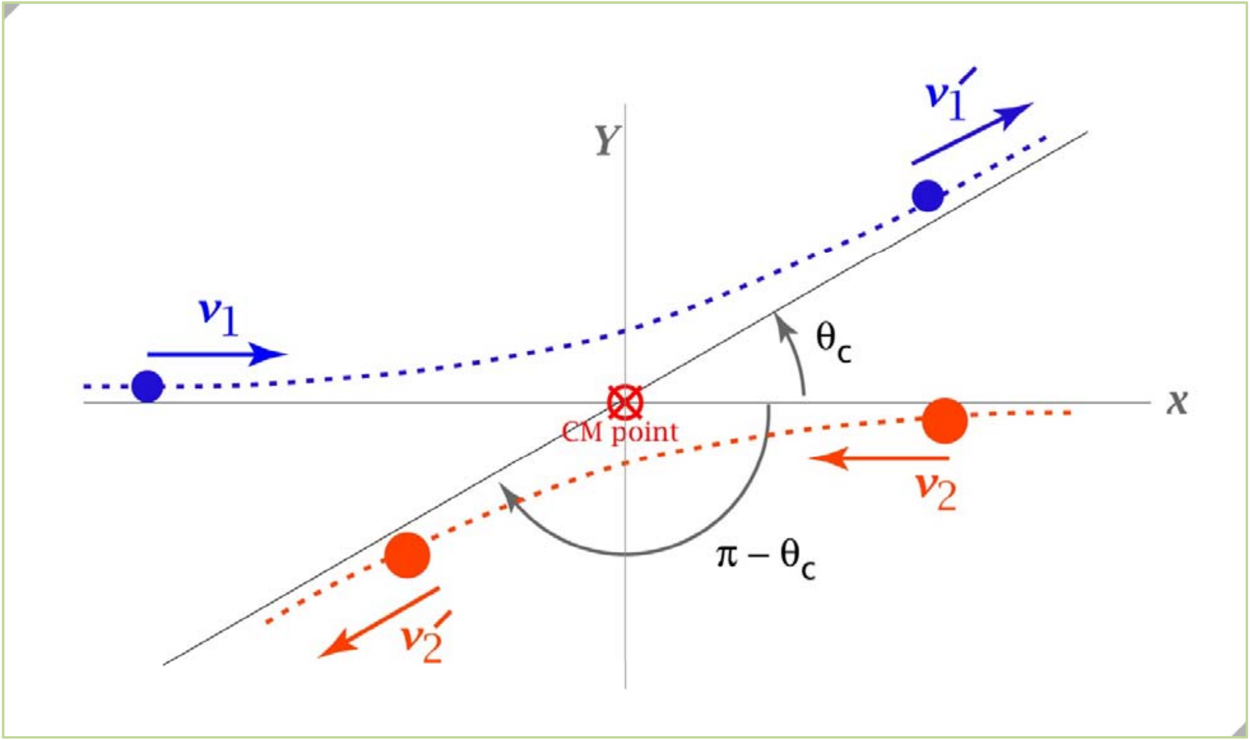
Thus, $\boldsymbol{v_1^2 = v_1'^2 + v_2'^2 + 2\, v_1' \cdot v_2'}$

But conservation of kinetic energy implies $\boldsymbol{v_1^2 = v_1'^2 + v_2'^2}$.

Thus, $\boldsymbol{v_1' \cdot v_2' = 0}$. Since ***the dot product is 0***, the vectors are perpendicular; i.e., the angle between the vectors is 90 degrees. (*Pool players know this empirically*!)

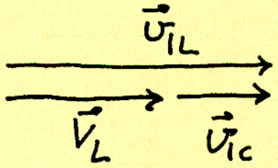
Elastic Collisions in the Center of Mass frame

An elastic collision in the center of mass frame is particularly simple: **First,** the angle between the outgoing particles is 180 degrees; i.e., if the scattering angle is θ then the recoil angle is $\pi - \theta$. **Second,** the final speeds are equal to the initial speeds.
[[Proof: Because with these final velocities, the total momentum is zero and the total kinetic energy is constant; that's obvious from the diagram.]]

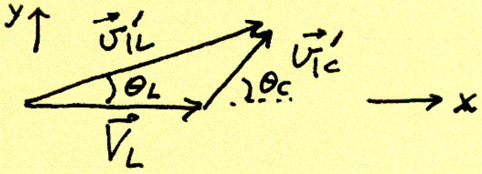


Energies and Scattering Angles
(Compare LAB and CM)

Before coll.



After coll.



$$V_L = \frac{m_1}{m_1+m_2} v_{1L}$$

$$v_{1c} = \frac{m_2}{m_1+m_2} v_{1L}$$

$$\vec{v}_{1L}' = \vec{V}_L + \vec{v}_{1c}'$$

$$= V_L \hat{i} + v_{1c}' (\hat{i} \cos\theta_c + \hat{j} \sin\theta_c)$$

where $v_{1c}' = v_{1c} = \frac{m_2}{m_1+m_2} v_{1L}$

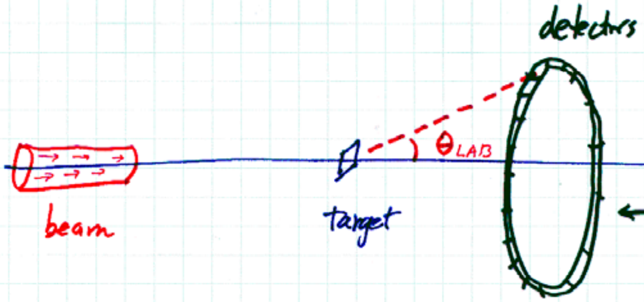
- $$\tan\theta_L = \frac{(v_{1L}')_y}{(v_{1L}')_x} = \frac{v_{1c}' \sin\theta_c}{V_L + v_{1c}' \cos\theta_c}$$

$$\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + (m_1/m_2)}$$

- $$K_{1L} = \frac{1}{2} m_1 v_{1L}^2 = \frac{m_1}{2} \left(\frac{m_1+m_2}{m_2} v_{1c} \right)^2$$

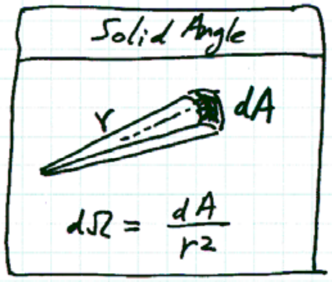
$$K_{1L} = \left(\frac{m_1+m_2}{m_2} \right)^2 K_{1c}$$

The scattering cross section



$\leftarrow d\Omega = 2\pi \sin\theta_{LaL}\, d\theta$

$$\frac{d\sigma}{d\Omega} = \frac{1}{I}\,\frac{dN}{d\Omega}$$


Solid Angle

$$d\Omega = \frac{dA}{r^2}$$

Calculate the cross section in the center of mass frame $\longrightarrow$ the equivalent 1-body problem

classical mechanics is deterministic $\Rightarrow$ $\theta_c(b)$.

$$\left(\frac{d\sigma}{d\Omega'}\right)_{CM} = \frac{b}{\sin\theta_c}\left|\frac{db}{d\theta_c}\right|$$

Proof: $2\pi b\,\delta b\, I = dN = I\left(\frac{d\sigma}{d\Omega'}\right) 2\pi \sin\theta_c\, \delta\theta_c$

$\therefore \left(\frac{d\sigma}{d\Omega'}\right) =$ as claimed.

The cross section in the lab frame of reference is $\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \frac{\sin\theta_c}{\sin\theta_L}\,\frac{d\theta_c}{d\theta_L}$ w/ $\theta_c \rightarrow \theta_L$

Dynamics for a System of Particles

3d – Transfer of Momentum or Mass

Review

(1) Dynamics of a single particle

$$m\frac{d\vec{v}}{dt} = \vec{F} \quad \text{and} \quad \frac{d\vec{x}}{dt} = \vec{v}$$

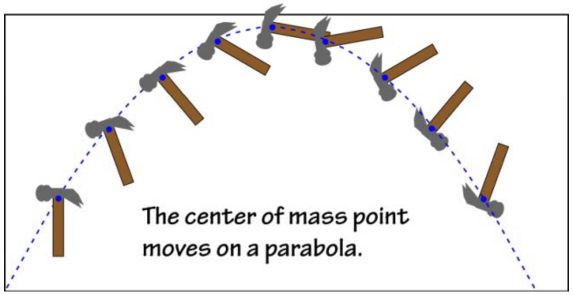
Or, we can write

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{and} \quad \frac{d\vec{x}}{dt} = \vec{v}$$

$$(\vec{p} = m\vec{v})$$

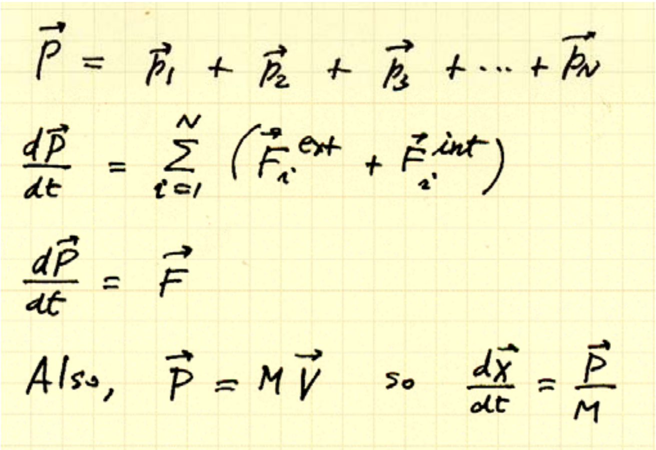
Mass does not change for a single particle.

(2) Dynamics of an extended object



The center of mass point moves on a parabola.

Imagine the object subdivided into many small parts.



$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots + \vec{p}_N$$

$$\frac{d\vec{P}}{dt} = \sum_{i=1}^{N} \left(\vec{F}_i^{ext} + \vec{F}_i^{int}\right)$$

$$\frac{d\vec{P}}{dt} = \vec{F}$$

Also, $\vec{P} = M\vec{V}$ so $\frac{d\vec{X}}{dt} = \frac{\vec{P}}{M}$

The center of mass motion is just like a *particle* !

Dynamics of many particles

(3) ***Dynamics of many particles***

Unlike a solid object, where strong internal forces hold the structure constant, a system may have internal motions.

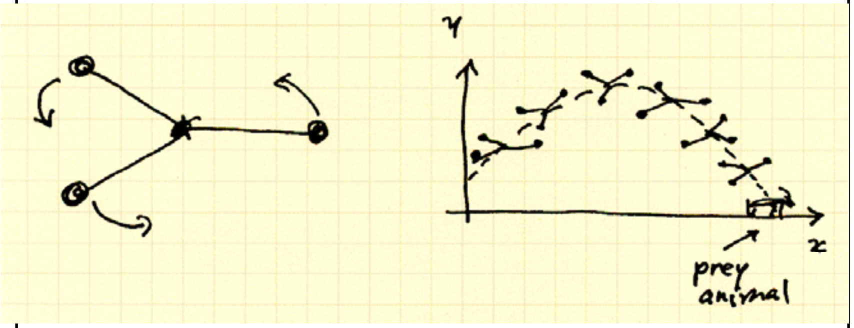
Example: Collisions

Still, the center of mass of the system moves as a particle

$$\frac{d\vec{P}}{dt} = \vec{F} \quad \text{and} \quad \frac{d\vec{X}}{dt} = \frac{\vec{P}}{M}$$

(P = total momentum; F = sum of external forces, M = total mass, X = center of mass position)

Example. The bola (a gaucho's hunting weapon)



The center of mass moves as a single particle projectile

$$\frac{d\vec{P}}{dt} = M\vec{g} \quad \text{and} \quad \frac{d\vec{X}}{dt} = \frac{\vec{P}}{M}$$

i.e., $d^2\mathbf{X}/dt^2 = \mathbf{g}$.

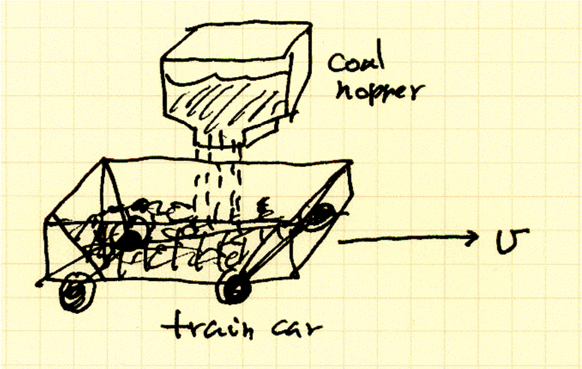
Transfer of Momentum or Mass

Next we'll consider systems where momentum or mass is transferred from one part of the system to another.

<u>This is another aspect of the dynamics of a system of particles</u>.

Example: Rockets

<u>Example.</u> Loading coal into a train car.

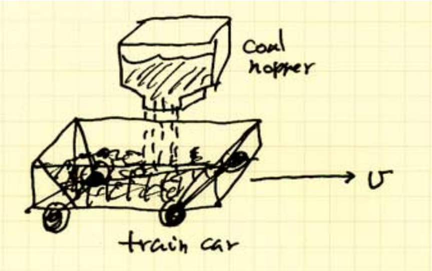


F = the ***external*** force on the car.

There are also ***internal*** forces when the coal lands in the car. Assume that the coal stops when it lands in the car (does not bounce around like a rubber ball). The dynamics is like *inelastic collisions*, occurring continuously in time.

Transfer of mass – to the car

Transfer of momentum – to the coal



Loading coal into a train car

First Case. F = 0 and v(0)= v_0. Determine v(t).
The dynamical system consists of the train car $\{M_{car}\}$ plus the coal $\{M_{coal}(t)\}$.

$$P = M_{car}\, v + M_{coal}\, v$$

Assume $M_{coal} = \mu t$ where $\mu = \frac{dM_c}{dt}$; i.e., constant rate

$$\frac{dP}{dt} = (M_{car} + M_{coal})\frac{dv}{dt} + \frac{dM_{coal}}{dt} v$$

$$= (M_c + \mu t)\frac{dv}{dt} + \mu v$$

$$\frac{dP}{dt} = F = 0$$

$$\frac{dv}{dt} = \frac{-\mu v}{M_c + \mu t}$$

Solution.

Velocity

$$\frac{dv}{v} = \frac{-\mu\, dt}{M_c + \mu t}$$

$$\ln v - \ln v_0 = -\ln(M_c + \mu t) + \ln(M_c)$$

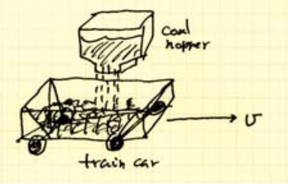
$$v = \frac{M_c v_0}{M_c + \mu t}$$

Distance

$$D = \int_0^t v\, dt = \frac{M_c v_0}{\mu} \ln \frac{M_c + \mu t}{M_c}$$

Acceleration

$$a = \frac{dv}{dt} = \frac{-\mu v_0 M_c}{(M_c + \mu t)^2}$$



Loading coal into a train car

Second Case. F = F_0 (a constant external force) and $v(0) = v_0$. Determine v(t).

Equation of motion

$$\frac{dP}{dt} = (M_c + \mu t)\frac{dv}{dt} + \mu v$$

$$\frac{dP}{dt} = F_0$$

$$(M_c + \mu t)\frac{dv}{dt} = F_0 - \mu v$$

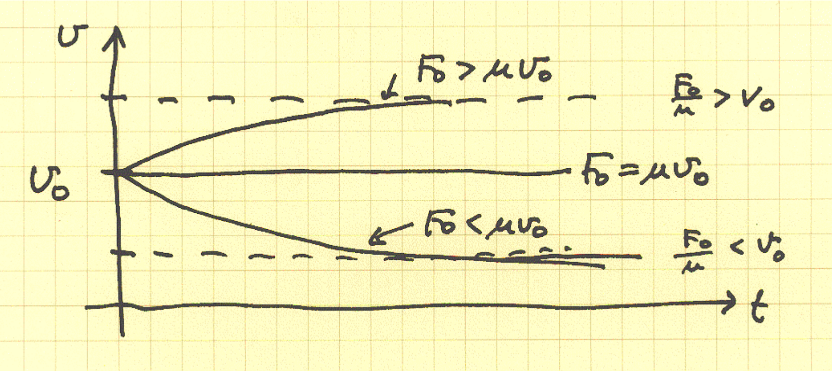
$$\frac{dv}{F_0 - \mu v} = \frac{dt}{M_c + \mu t}$$

$$\frac{1}{\mu}\left[\ln(F_0 - \mu v_0) - \ln(F_0 - \mu v)\right] = \frac{1}{\mu}\left[\ln(M_c + \mu t) - \ln M_c\right]$$

Solution.

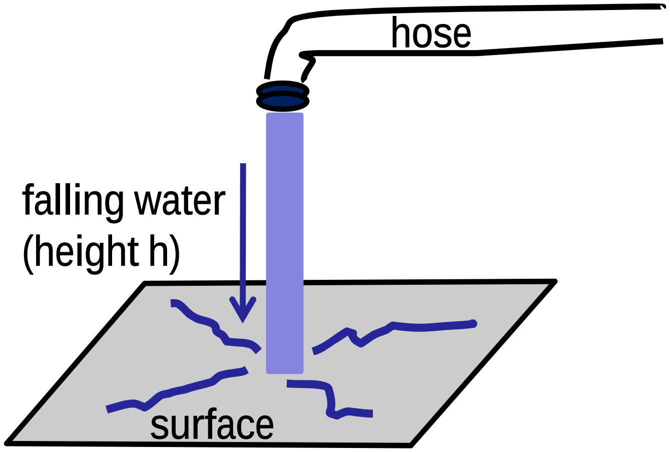
$$\frac{F_0 - \mu v}{F_0 - \mu v_0} = \frac{M_c}{M_c + \mu t}$$

$$v = \frac{M_c v_0 + F_0 t}{M_c + \mu t}$$



The force due to falling water

Example. Water flows in the hose at velocity v_0. The diameter of the hose is d. Calculate the force on the surface.



Solution.

Consider a small time interval dt.
Calculate the change of momentum of the water that hits the surface during that time.

mass: $dm = r\, A\, v_0\, dt$

velocity at the surface: $v_S = \sqrt{v_0^2 + 2\, g\, h}$

change of momentum $dP = dm\, v_S$ (approx.)

normal force: $N = dP\, /\, dt = r\, A\, v_0\, v_S$

Numerical Example.

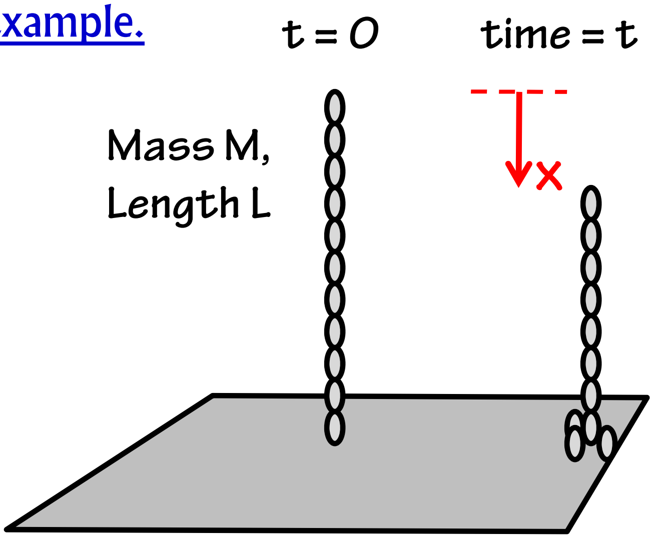
$d = 2.54$ cm , $v_0 = 30.48$ cm/s , $h = 30.48$ cm

$A = 5.07 \times 10^{-4}$ m^2 ; $v_S = 2.46$ m/s ;
($r = 1000$ kg/m^3)

$N = 0.380$ newton (= force on the surface due to momentum transfer)

The Falling Chain

Example.



Calculate the force on the surface.

Solution. Treat the chain as a continuum.

Free fall: $v = g\,t$ and $x = \frac{1}{2} g\,t^2$

$P(t) = \rho_{len}\,(L - x)\;v$ where $\rho_{len} = M/L$

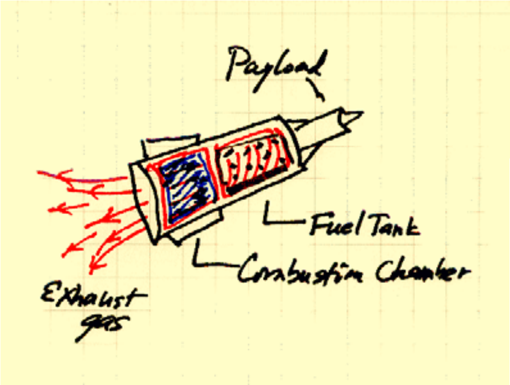
$dP/dt = F_{ext} = Mg - N$

approx.; assuming the links don't bounce

I'll leave the rest of the calculation as an exercise.

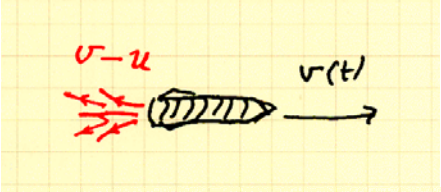
Answer: When the top of the chain hits the table, i.e., for $x = L$, the force on the surface is

$$F_{surf} = -N = 3\,Mg.$$



Dynamics for a System of Particles

3-5: Rockets



Combustion releases chemical energy, which is converted into kinetic energy of the exhaust gases. Total momentum …

$$P = P_{\text{rocket}} + P_{\text{exhaust}}$$

$$\frac{dP}{dt} = F_{\text{external}}$$

These are the principles of dynamics for a system of particles (rocket & gas).

Two parameters of the exhaust gas

μ = mass rate (kg/s)

u = relative speed (m/s)

Example 1: An isolated rocket with constant μ and u (in free space).

$$F_{\text{external}} = 0$$

isolated rocket

$$\frac{dP}{dt} = 0$$

For a short time interval δt, calculate δP

$$\delta P = P(t+\delta t) - P(t)$$

Let m(t) = mass of the rocket (and enclosed fuel) at time t; let v(t) = the velocity of the rocket at time t. Then

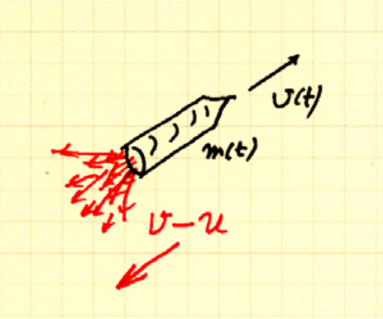
$$P(t) = m(t)\ v(t)$$

the rocket at time t

Rocket in Free Space

$P(t) = m(t)\ v(t)$ rocket and enclosed gas at t

$P(t+\delta t) = m(t+\delta t)\, v(t+\delta t) + (v-u)\, \mu\, \delta t$



momentum of the gas expelled during time δt

$$\delta P = m(t+\delta t)v(t+\delta t)+(v-u)\mu\, \delta t - m(t)v(t)$$

$$\delta P = (m-\mu\, \delta t)\,(v + \delta v) + (v-u)\, \mu\, \delta t - m\, v$$

$$\delta P = m\, \delta v - \mu\, u\, \delta t \quad + \ O\,(\delta t)^2$$

Important: We can neglect that in the limit δt → 0.

$$\frac{dP}{dt} = 0 \Rightarrow m\frac{dv}{dt} - \mu\, u = 0$$

Also,

$$m(t) = m_0 - \mu\, t$$

So the equation of motion is

$$(m_0 - \mu\, t\,)\frac{dv}{dt} = \mu\, u$$

Separation of variables

$$dv = \frac{\mu u}{m_0 - \mu t}\, dt$$

$$\int_{v_0}^{v} dv = \int_0^t \frac{\mu u}{m_0 - \mu t}\, dt$$

$$v - v_0 = \mu u \left(\frac{-1}{\mu}\right) \ln (m_0 - \mu t)\Big]_0^t$$

$$= u \left\{ -\ln (m_0 - \mu t) + \ln m_0 \right\}$$

$$= u \ln \frac{m_0}{m_0 - \mu t}$$

$$v(t) = v_0 + u \ln \frac{m_0}{m_0 - \mu t}$$

Rocket in Free Space

Equation of motion:

$$(m_0 - \mu\, t)\,\frac{dv}{dt} = \mu\, u$$

Solution:

$$v = v_0 + u \ln \frac{m_0}{m_0 - \mu\, t}$$

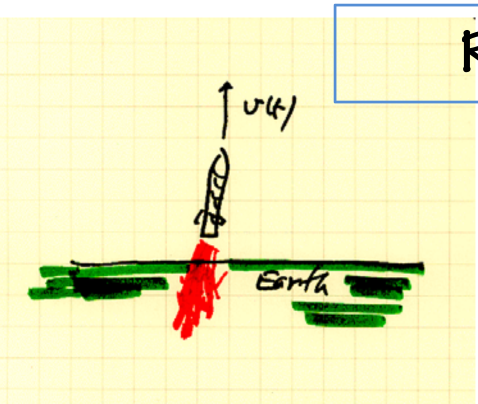
and $\frac{dm}{dt} = -\mu$

masses	
m_R	Empty Rocket
m_F	Initial Fuel
m_0	$m_R + m_F$
m	$m_0 - \mu\, t$

Exercise. Show, for any time dependence of the mass rate $\mu(t)$, but still assuming that u is constant (independent of time),

$$v(t) = v_0 + u \ln \frac{m_0}{m(t)}$$

Rocket at the surface of the Earth



Example 2: A rocket with constant μ and u, at the surface of the Earth.

$$F_{external} = -mg$$

$$\frac{dP}{dt} = -mg$$

For a short time interval δt, calculate δP

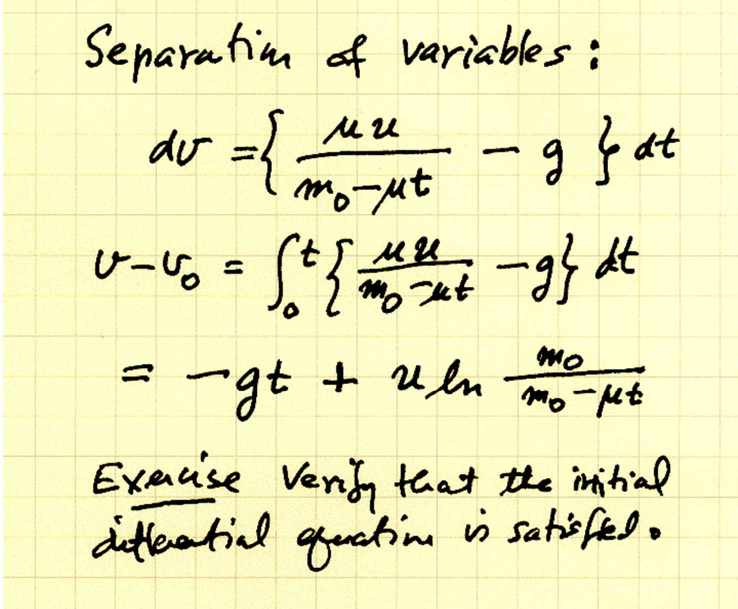
$$\delta P = P(t+\delta t) - P(t)$$

Same as before,

$$\frac{dP}{dt} = m\frac{dv}{dt} - \mu u$$

Equation of motion:

$$m\frac{dv}{dt} = \mu u - m g$$



Separation of variables:

$$dv = \left\{ \frac{\mu u}{m_0 - \mu t} - g \right\} dt$$

$$v - v_0 = \int_0^t \left\{ \frac{\mu u}{m_0 - \mu t} - g \right\} dt$$

$$= -gt + u \ln \frac{m_0}{m_0 - \mu t}$$

Exercise Verify that the initial differential equation is satisfied.

Solution:

$$v = v_0 - g t + u \ln \frac{m_0}{m_0 - \mu t}$$

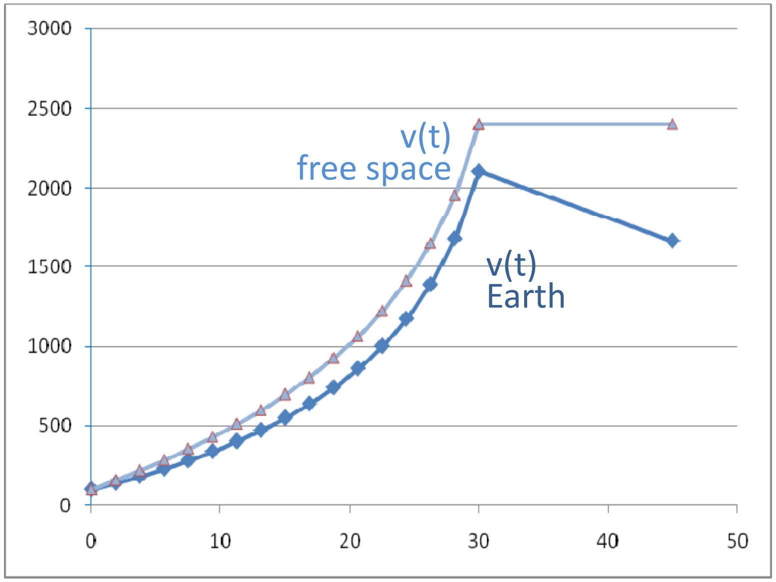
Rocket at the surface of the Earth

Solution:

$$v = v_0 - g\,t + u \ln \frac{m_0}{m_0 - \mu\,t}$$

Parameters: $v_0 = 100\text{m/s}$

m_R	m_F	$t_{burnout}$	u
100kg	900kg	30s	1000m/s



Exercise:

What is the condition for *take-off;*
i.e., upward acceleration (a > 0) at t = 0?

Answer:

$a = dv/dt = -g + u\,\mu/(m_0 - \mu t)$

a(0) > 0 requires $\boxed{\mu\,u > m_0\,g}$

In words, the thrust must be > the weight.

Exercise:

What is the height at burnout?

Answer:

$$H_b = \int_0^{t_b} v(t)\ dt\ = \int_0^{m_F/\mu} \left(-g\,t\ +\ u \ln \frac{m_0}{m_0 - \mu\,t}\right) dt$$

(I'll leave the calculation as an exercise.)

Note: H_b is not the maximum height. The rocket is still moving upward at burnout. It reaches the maximum height when v = 0.