Acoustics of the cylindrical resonator

The sound field generated in a cylindrical resonator of length L and radius a is given by the wave equation for the pressure p

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

where c is the speed of sound and t is time. The solutions are of the form

$$p(r,\theta,z,t) = \Theta(\theta)R(r)Z(z)e^{i\omega t}$$

where z represents the direction along the cylinder axis, r is the radial direction and θ is the azimuthal angle and

$$\Theta(\theta) = e^{im\theta}$$
$$R(r) = b_1 J_m(kr_r) + b_2 Y_m(kr_r)$$
$$Z(z) = b_3 \sin(kz_z) + b_4 \cos(kz_z)$$

Where J_m and Y_m are Bessel functions of the 1st and 2nd kind, respectively. If the source field is axisymmetric there will be no θ-dependence and m= 0. If we assume the end walls of the cavity at z = 0, L and at r =a are rigid, then the axial and radial particle velocities are zero (nodes). Then at the boundaries we have $\partial Z/\partial z = \partial R/\partial r = 0$. Applying these boundary conditions, we find for the axial modes, $k_q^z = q\frac{\pi}{L}(q = 1,2,3,...)$ and for the radial modes, $k_{nm}^r = \frac{j_{nm}}{a}$, where $j_{mn} = \frac{\partial J_m(k_{nm}^r r)}{\partial r} = 0$, i.e. the nth stationary value of j_{mn}. The term j_{0n} has values 3.84, 7.02, 10. 18, 13.32, ... for n = 1, 2, 3, 4,... [Note that these boundary conditions do not apply to a cavity driven from the ends since the ends will be at antinodes.]

Since $k_z^2 + k_r^2 = k^2$ and $c = \omega / k$ the resonance frequencies (in Hz) are given by

$$f_{nq}^2 = \left(\frac{c}{2\pi}\right)^2\left[\left(\frac{j_{0n}}{a}\right)^2 + \left(\frac{q\pi}{L}\right)^2\right].$$

In the limit L<<a, one expects to see only plane-wave-like axial modes at low frequencies. By measuring the resonances for a series of q-values one can accurately determine the speed of sound c. The mode index q is the number of half-wavelengths that can be fit into the cavity along the z-axis at resonance.

Using a spreadsheet, calculate the expected resonance frequencies for the experimental cavity for at room temperature for air, N_2, and He gases. Identify the frequency of the lowest Bessel mode. Also calculate the resonant frequencies for 1st sound in liquid He I at 4.2K and for 2nd sound in He II at 1.8 K.

Frequency response of a cavity driven near resonance

Now we consider the response of a cavity driven by a periodic field. In analogy with a one-dimensional harmonic oscillator, it is useful to see how the amplitude and phase of the system varies in the vicinity of a resonance, e.g., for an axial mode of the cylindrical cavity. Let z represent the displacement of a microphone. The cavity is driven at the transmitter by the field

$$z_t(\omega) = z_0 \sin \omega t$$

whereas the receiver sees the field

$$z_r(\omega) = z_0 A(\omega) \sin(\omega t - \delta).$$

This says that the receiver sees a signal with frequency-dependent amplitude that is shifted in phase by δ. Near a resonance,

$$A(\omega) = \frac{\omega^2}{\left[(\omega_0{}^2 - \omega^2)^2 + (\omega\Gamma)^2\right]^{1/2}}$$

where ω_0 is the bare cavity resonance frequency and Γ is the damping constant. The phase shift δ is frequency dependent, with

$$\tan\delta = \frac{Q^{-1}}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)} = \frac{\Gamma\omega_0}{\omega_o{}^2 - \omega^2}$$

where we have defined the quality factor $Q = \omega_0 / \Gamma$. Note that as $\omega \to \omega_0, \tan\delta \to \infty, \delta \to \pi/2$.

The signal at the receiver can be written in the following form:

$$z_r(\omega) = z_0 A(\omega) \sin(\omega t - \delta) = z_0 A(\omega)\left[\sin \omega t \cos\delta - \cos \omega t \sin\delta\right]$$

The lock-in reads each of the two orthogonal components independently as X and Y. The actual phase is arbitrary. The best procedure is to find the resonance using R (amplitude) since it is independent of phase. Then push the AutoSet button to set Y to zero giving the maximum value for X (=R).