

Appendix B

Dealing with uncertainty

B.1 Overview

- An uncertainty is always a positive number $\delta x > 0$.
- If you measure x with a device that has a precision of u , then δx is at least as large as u .
- **Fractional uncertainty:**
 - If the fractional uncertainty of x is 5%, then $\delta x = 0.05x$.
 - If the uncertainty in x is δx , then the fractional uncertainty in x is $\delta x/x$.

- **Propagation of uncertainty:**

- If $z = x + y$ or if $z = x - y$, then

$$\delta z = \delta x + \delta y. \quad (\text{B.1})$$

- If $z = xy$ or if $z = x/y$, then

$$\frac{\delta z}{|z|} = \frac{\delta x}{|x|} + \frac{\delta y}{|y|} \quad (\text{B.2})$$

- For $f = x^n y^m z^p$, where n , m , and p are exact,

$$\frac{\delta f}{f} = n \frac{\delta x}{x} + m \frac{\delta y}{y} + p \frac{\delta z}{z} \quad (\text{B.3})$$

– For an arbitrary function $f(x)$,

$$\delta f(x) = |f(x + \delta x) - f(x)| \quad (\text{B.4})$$

as long as $\delta x \ll x$.

- **Percent error.** If d is data and e is the expected value, the difference, D , is

$$D = d - e. \quad (\text{B.5})$$

The percent error is given by

$$\% \text{ error} = \frac{D}{e} \times 100\% \quad (\text{B.6})$$

They are compatible if $|d - e| < \delta d + \delta e$, that is they are compatible when their difference is equal to 0 with the uncertainty of the difference.

B.2 Concise notation of uncertainty

If, for example, $y = 1\,234.567\,89\text{ U}$ and $\delta y = 0.000\,11\text{ U}$, where U is the unit of y , then $y = (1\,234.567\,89 \pm 0.000\,11)\text{ U}$. A more concise form of this expression, and one that is in common use, is $y = 1\,234.567\,89(11)\text{ U}$, where it is understood that the number in parentheses is the numerical value of the standard uncertainty referring to the corresponding last digits of the quoted result. This explanation is from Ref. (Mohr 2011).

B.3 Significant figures

There is no actual information carried by figures which represent values much smaller than the uncertainty of a measurement. For example, if you do a calculation and your calculator, or Excel, gives you $x = 12.3456789$, but when you calculate the uncertainty, you get $\delta x = 0.01234$, according to the previous section you would naturally write this as

$$12.346(12).$$

This is because there is hardly ever justification for reporting an uncertainty to more than 2 figures (so that 0.01234 should be reported as just 0.012).

Therefore the decimal places in x beyond the reported uncertainty are obviously carrying information corresponding to a tiny fraction of your actual knowledge of x (i.e. much smaller than δx). Thus $12.3456789 \rightarrow 12.346$ since the last reported position in the uncertainty is 0.012, 3 figures to the right of the decimal point; and in the reported x value, the trailing 56789 would be rounded up to 6.

B.4 Using uncertainties to compare data and expectations

Simple Measurements: The smallest division estimate

Suppose you use a meter stick ruled in centimeters and millimeters, and you are asked to measure the length of a rod and obtain the result $L_0 = 5.73$ cm, seen in Fig. B.1a. A good estimate of the *uncertainty* here is **half the smallest division on the scale**, 0.05 cm. Thus, the length of the rod would be specified as

$$L_0 = 5.73 \pm 0.05 \text{ cm.} \quad (\text{B.7})$$

This says that you are very confident that the length of the rod falls in the range $5.73 \text{ cm} - 0.05 \text{ cm}$ to $5.73 \text{ cm} + 0.05 \text{ cm}$ — that is, the length falls in the range of 5.68 cm to 5.78 cm, as in Fig. B.1b.

Manufacturer's tolerance

Suppose I purchase a nominally 100Ω resistor from a manufacturer. It has a gold band on it which signifies a 5% tolerance. What does this mean? The tolerance means $\delta R/R = 0.05 = 5\%$, that is, the fractional uncertainty. Thus, $\delta R = R \times 0.05 = 5 \Omega$. We write this as

$$R = R_{\text{nominal}} \pm \delta R = 100 \pm 5 \Omega. \quad (\text{B.8})$$

It says that the company certifies that the true resistance R lies between 95 and 105Ω . That is, $95 \leq R \leq 105 \Omega$. The company tests all of its resistors and if they fall outside of the tolerance limits the resistors are discarded. If your resistor is measured to be outside of the limits, either (a) the manufacturer made a mistake, (b) you made a mistake, or (c) the manufacturer shipped the correct value but something happened to the resistor that caused its value to change.

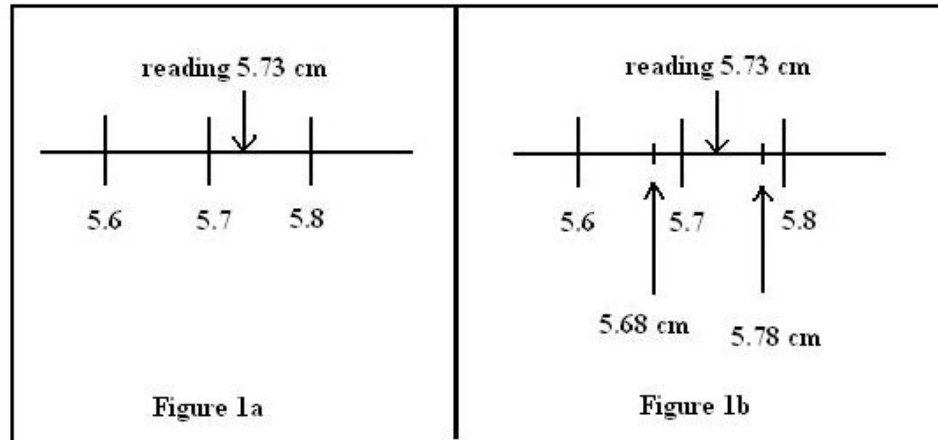


Figure B.1: Measuring a length with a ruler. If the value is read as in (a), then a reasonable uncertainty is shown in (b).

Reading a digital meter

Suppose I measure the voltage across a resistor using a digital multimeter. The display says 7.45 V and doesn't change as I watch it. The general rule is that **the uncertainty is half of the value of the least significant digit**. This value is 0.01 V, so the uncertainty is half of it — 0.005 V. Here's why: The meter can only display two digits to the right of the decimal, so it must round off additional digits. So if the true value is between 7.445 V and 7.454 V, the display will get rounded to 7.45 V. Thus the average value and its uncertainty can be written as 7.45 ± 0.005 V.

When you record this in your notebook, be sure to write 7.45 V. Not 7.450 V. Writing 7.450 V means that the uncertainty is 0.0005 V.

Note that in this example we assumed that the meter reading is **steady**. If, instead, the meter reading is fluctuating, then the situation is different. Now you need to estimate the range over which the display is fluctuating, then estimate the average value. If the display is fluctuating between 5.4 and 5.8 V, you would record your reading as 5.6 ± 0.2 V. The uncertainty due to the noisy reading is much larger than your ability to read the last digit on the display, so you record the larger error.

Using uncertainties in calculations

We need to combine uncertainties so that the error bars almost certainly include the true value.

Adding and subtracting

Let's look at the most basic case. We measure x and y and want to find the error in z .

If $z = x + y$, then

$$\delta z = \delta x + \delta y \quad (\text{B.9})$$

If $z = x - y$, then

$$\delta z = \delta x + \delta y \quad (\text{B.10})$$

Note that the uncertainty for subtracting has exactly the same form as for adding.

The most important errors are simply the biggest ones, since they impact the precision of your result the most.

Example:

$$(7 \pm 1 \text{ kg}) - (5 \pm 1 \text{ kg}) = 2 \pm 2 \text{ kg} \quad (\text{B.11})$$

Multiplying and dividing

If $a = bc$, then

$$\frac{\delta a}{a} = \frac{\delta b}{b} + \frac{\delta c}{c} \quad (\text{B.12})$$

For dividing, if $w = x/y$, the rule is the same as for multiplication;

$$\frac{\delta w}{w} = \frac{\delta x}{x} + \frac{\delta y}{y} \quad (\text{B.13})$$

It is simplest to just remember the single boxed rule, Eq. [B.12](#), for multiplication and division.

If the expression contains a constant, the uncertainty of that constant is zero.

The most important errors in multiplication and division are the largest *fractional* errors, not absolute errors. This makes sense if you consider that

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b and c need not have the same units — there is no way to compare the absolute sizes of quantities with different units.

Example:

$$\begin{aligned}V &= IR \\I &= 7 \pm 1 \text{ mA} \\R &= 20 \pm 2 \Omega\end{aligned}\tag{B.14}$$

$$V = 140 \text{ (mA} \cdot \Omega) = 140 \text{ mV} = 0.14 \text{ V}$$

The uncertainty is given by

$$\begin{aligned}\frac{\delta V}{V} &= \frac{\delta I}{I} + \frac{\delta R}{R} \\&= \frac{1 \text{ mA}}{7 \text{ mA}} + \frac{2 \Omega}{20 \Omega} \\&= 0.24\end{aligned}\tag{B.15}$$

$$\begin{aligned}\delta V &= 0.24 \times (0.14 \text{ V}) \\&= 34 \text{ mV}\end{aligned}$$

Our formula for multiplication indicates that multiplying by a perfectly known constant has no effect on the *fractional* error of a quantity. For example, the speed of light in vacuum, c , is 299 792 458 m/s with no uncertainty.¹ If we measure the time it takes for light to travel as 12 ± 1 s, then we can find the distance that it traveled.

$$\begin{aligned}c &= 299\,792\,458 \text{ m/s} & t &= 12 \pm 1.5 \text{ s} \\d &= ct \\d &= (299\,792\,458 \text{ m/s}) \times (12 \text{ s}) \\d &= 3\,597\,509\,496 \text{ m}\end{aligned}\tag{B.16}$$

¹This is because the meter is defined as the distance light travels in $1/299\,792\,458$ seconds in a vacuum.

B.4. Using uncertainties to compare data and expectations

The uncertainty is given by

$$\begin{aligned}\frac{\delta d}{d} &= \frac{\delta c}{c} + \frac{\delta t}{t} \\ &= \frac{0 \text{ m/s}}{299\,792\,458 \text{ m/s}} + \frac{1.5 \text{ s}}{12 \text{ s}} \\ &= \frac{1.5}{12}\end{aligned}\tag{B.17}$$

and thus $d = 3\,597\,509\,496 \pm 449\,688\,687 \text{ m}$. Note that the value of the speed of light did not matter in the calculation of the fractional uncertainty, since it was multiplied by its zero uncertainty.

The uncertainty $\delta d = d \frac{\delta t}{t} = ct \frac{\delta t}{t} = c\delta t$.

So, δd is just the constant c times δt .

Multiples

If $f = cx + dy + gz$, where c , d , and g are positive or negative constants, then from the multiplication rule, we find that

$$\begin{aligned}\delta(cx) &= |c\delta x| \\ \delta(dy) &= |d\delta y| \\ \delta(gz) &= |g\delta z|\end{aligned}\tag{B.18}$$

From the addition rule,

$$\delta f = |c\delta x| + |d\delta y| + |g\delta z|\tag{B.19}$$

Powers

If $f = x^p y^q z^r$, where p , q , and r are positive or negative constants,

$$\frac{\delta f}{f} = \frac{\delta(x^p)}{x^p} + \frac{\delta(y^q)}{y^q} + \frac{\delta(z^r)}{z^r}\tag{B.20}$$

and thus

$$\frac{\delta f}{f} = \left| p \frac{\delta x}{x} \right| + \left| q \frac{\delta y}{y} \right| + \left| r \frac{\delta z}{z} \right| \quad (\text{B.21})$$

General formula

Suppose we want to calculate $f(x)$, a function of x , which has uncertainty δx . What is the uncertainty in the calculated value f ? We simply calculate f at x , and again at $x' = x + \delta x$, then take the absolute value of the difference:

$$\delta f = |f(x') - f(x)|, \text{ where } x' = x + \delta x. \quad (\text{B.22})$$

For example, if $f(x) = \sin x$, and $x = 30 \pm 1^\circ$, then

$$\begin{aligned} \delta f &= |\sin(31^\circ) - \sin(30^\circ)| \\ &= |0.515 - 0.500| \\ &= 0.015 \end{aligned} \quad (\text{B.23})$$

What happens when there is more than one variable? We do the calculation for each variable separately and combine the resulting uncertainties:

$$\delta f(x, y) = |f(x + \delta x, y) - f(x, y)| + |f(x, y + \delta y) - f(x, y)| \quad (\text{B.24})$$

When are errors negligible?

Errors are only negligible in comparison to something else and in the context of a particular calculation. So it's hard to give general rules, but easier for specific cases. Here's an example of how to think about this question.

You measure a long thin ribbon (that is, something rectangular). Its length is 10 ± 0.02 m, and its width is 2 ± 0.1 cm. Which uncertainty is more important? The answer depends on what you want to calculate.

First, imagine that you are feeling festive and want to border the ribbon with glitter. To know how much glitter you'll need, you must find the length of the perimeter P of the rectangle formed by the ribbon. The perimeter is given by

$$P = 2(L + W). \quad (\text{B.25})$$

We apply our addition rule:

$$\delta P = 2(\delta L + \delta W). \quad (\text{B.26})$$

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Now, $\delta L = 0.02 \text{ m}$ and $\delta W = 0.1 \text{ cm} = 0.001 \text{ m}$. Since δL is 20 times the size of δW , we can neglect δW — that is, ignore it. Note that we had to put δL and δW into the same units to compare them: 0.02 m is much larger than 0.2 cm .

Now, imagine that instead of just the border, you want to cover the entire area of one side of the ribbon with glitter. For this you need to find the area, A , of the ribbon.

$$A = LW \quad (\text{B.27})$$

The multiplication rule gives

$$\begin{aligned} \frac{\delta A}{A} &= \frac{\delta L}{L} + \frac{\delta W}{W} \\ &= \frac{0.02 \text{ m}}{10 \text{ m}} + \frac{0.1 \text{ cm}}{2.0 \text{ cm}} \\ &= 0.002 + 0.05 \end{aligned} \quad (\text{B.28})$$

In this case, the uncertainty due to δL is negligible compared to that from δW , the opposite conclusion as for the perimeter calculation! That's because we are multiplying and now need to compare not δL vs δW , but instead $\frac{\delta L}{L}$ vs $\frac{\delta W}{W}$.

Using uncertainties to compare data and expectations

One important question is whether your results agree with what is expected. Let's denote the data by d and the expected value by e . The ideal situation would be $d = e$, or $d - e = 0$. We'll use D to denote the difference between two quantities:

$$D = d - e \quad (\text{B.29})$$

The standard form for comparison is always (result) $-$ (expected), so that your difference D will be negative if your value is lower than expected, and positive if it is higher than expected.

This comparison must take into account the uncertainty in the observation, and perhaps, in the expected value as well. The data value is $d \pm \delta d$ and the expected value is $e \pm \delta e$. Using the addition/subtraction rule for uncertainties, the uncertainty in $D = d - e$ is just

$$\delta D = \delta d + \delta e \quad (\text{B.30})$$

Our comparison becomes, “is zero within the uncertainties of the difference D ?” Which is the same thing as asking if

$$|D| \leq \delta D \quad (\text{B.31})$$

Eqs. B.30 and B.31 express in algebra the statement “ d and e are compatible if their error bars touch or overlap.” The combined length of the error bars is given by Eq. B.30. $|D|$ is the magnitude of the separation of d and e . The error bars will overlap (or touch) if d and e are separated by less than (or equal to) the combined length of their error bars, which is what Eq. B.31 says.

Example

Now we have all we need to do comparisons. For example, if we measure a length of 5.9 ± 0.1 cm and expect 6.1 ± 0.1 cm (measured by the TA), the difference is

$$\begin{aligned} D &= d - e \\ &= 5.9 \text{ cm} - 6.1 \text{ cm} \\ &= -0.2 \text{ cm} \end{aligned} \quad (\text{B.32})$$

while the uncertainty of that difference is

$$\begin{aligned} \delta D &= \delta d + \delta e \\ &= 0.1 \text{ cm} + 0.1 \text{ cm} \\ &= 0.2 \text{ cm} \end{aligned} \quad (\text{B.33})$$

We conclude that our measurement is indeed (barely) consistent with expectations. If we had instead measured 6.4 cm, we would not have been consistent.

A good form to display such comparisons is:

d [cm]	δd [cm]	e [cm]	δe [cm]	D [cm]	δD [cm]	compatible?
5.9	0.1	6.1	0.1	-0.2	0.2	YES
6.4	0.1	6.1	0.1	+0.3	0.2	NO
6.2	0.2	6.1	0.1	+0.1	0.3	YES
6.4	0.2	6.1	0.0	+0.3	0.2	NO

If only one comparison is to be made, your lab report might contain a sentence like the following: “The measured value was 6.4 ± 0.2 cm while the

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expected value was 6.10 ± 0.0 cm, so the difference is $+0.3 \pm 0.2$ cm which means that our measurement was close to, but not compatible with, what was expected.”