Problem 5

In the Heisenberg picture,

$$\Psi(\vec{x},t) = e^{iHt/\hbar}\, \underbrace{\Psi(\vec{x},0)}\, e^{-iHt/\hbar}$$

take this to be the field in the Schroedinger picture.

$$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar\, e^{iHt/\hbar}\, \frac{i}{\hbar}\left[H, \Psi_s(x)\right] e^{-iHt/\hbar}$$

$$= -\, e^{iHt/\hbar}\left[H, \Psi_s\right] e^{-iHt/\hbar}$$

The Hamiltonian is $\hat{H} = \hat{T} + \hat{V}$.

$\hat{T}$ term $\quad \hat{T} = \int d^3x'\, \hat{\Psi}^+(x')\, T(x')\, \Psi(x')$

$$[\hat{T}, \Psi(x)] = \int d^3x' \left[\Psi^+(x')\, T(x')\, \Psi(x'), \Psi(x)\right]$$

$$= \int d^3x' \Big[\Psi^+(x') \{T(x')\Psi(x'), \Psi(x)\}$$

$$- \{\Psi^+(x'), \Psi(x)\}\, T(x')\Psi(x') \Big]$$

$$= \int d^3x' \left[0 - \delta^3(\vec{x}-\vec{x}')\, T(x')\Psi(x') \right]$$

$$= -\, T(x)\Psi(x)$$

$\hat{V}$ term $\quad \hat{V} = \frac{1}{2}\int d^3x_1\, d^3x_2\, \Psi^+(x_1)\, \Psi^+(x_2)\, V(x_1,x_2)\, \Psi(x_2)\, \Psi(x_1)$

$$[\hat{V}, \Psi(x)] = \frac{1}{2}\int d^3x_1\, d^3x_2\, V(x_1,x_2) \left[\Psi^+(x_1)\Psi^+(x_2)\Psi(x_2)\Psi(x_1), \Psi(x)\right]$$

$$= \Psi^+(x_1)\Psi^+(x_2)\left[\Psi(x_2)\Psi(x_1), \Psi(x)\right]$$

$$+ \left[\Psi^+(x_1)\Psi^+(x_2), \Psi(x)\right]\Psi(x_2)\Psi(x_1)$$

$$= 0 + \Big[\Psi^+(x_1)\{\Psi^+(x_2), \Psi(x)\}$$

$$- \{\Psi^+(x_1), \Psi(x)\}\Psi^+(x_2) \Big] \Psi(x_2)\Psi(x_1)$$

$$= \delta^3(x-x_2)\, \Psi^+(x_1)\Psi(x_2)\Psi(x_1)$$

$$- \delta^3(x-x_1)\, \Psi^+(x_2)\Psi(x_2)\Psi(x_1)$$

$\therefore$

$$[\hat{V}, \Psi(x)] = \frac{1}{2}\int d^3x_1\, V(x_1, x)\, \Psi^+(x_1)\Psi(x)\Psi(x_1)$$

$$- \frac{1}{2}\int d^3x_2\, V(x, x_2)\, \Psi^+(x_2)\Psi(x_2)\, \Psi(x)$$

$$= -\int d^3x'\, V(x x')\, \Psi^+(x')\Psi(x')\, \Psi(x)$$

Thus

$$i\hbar \frac{\partial \Psi}{\partial t} = T(x)\Psi_H(x,t) + \int d^3x'\, V(x x')\, n_H(x',t)\, \Psi_H(x,t)$$

2 points

Problem 6 We have

$$E^{(1)} = -4\pi e^2 \frac{V}{(2\pi)^6} \int \frac{d^3q}{q^2} \int d^3P\, \theta(k_F - |\vec{P} + \vec{q}/2|)\, \theta(k_F - |\vec{P} - \vec{q}/2|)$$

First integral

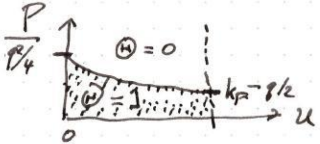
$$J = \int d^3P\; \theta(k_F - |P + \tfrac{1}{2}q|)\; \theta(k_F - |P - q/2|)$$

$$= \int_0^\infty P^2 dP \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\; \theta\left(k_F - \sqrt{P^2 + q^2/4 + Pq\cos\theta}\right) \theta\left(k_F - \sqrt{P^2 + q^2/4 - Pq\cos\theta}\right)$$

$$\int_0^{2\pi} d\phi = 2\pi \;;\quad \int_0^\pi \sin\theta\, d\theta = \int_{-1}^1 du = 2\int_0^1 du$$

(where $u = \cos\theta$)

$$J = 4\pi \int_0^\infty P^2 dP \int_0^1 du\; \theta\left(k_F - \sqrt{P^2 + q^2/4 + Pqu}\right)$$



$$P^2 + q^2/4 + Pqu < k_F^2$$

$$u = 0:\quad P < \sqrt{k_F^2 - q^2/4}$$

$$u = 1:\quad P < k_F - q/2$$

$$J = 4\pi \int_0^{k_F - q/2} P^2 dP + 4\pi \int_{k_F - q/2}^{\sqrt{k_F^2 - q^2/4}} P^2 dP \left(\frac{k_F^2 - P^2 - q^2/4}{Pq}\right)$$

Let $q = 2k_F x$; Note that $q < 2k_F$ so $x < 1$.

$$J = \frac{4\pi}{3}[k_F(1-x)]^3 + \frac{4\pi}{q}\Big[k_F^2(1-x^2)\tfrac{1}{2}\left(k_F^2(1-x^2) - k_F^2(1-x)^2\right) - \tfrac{1}{4}\left(k_F^4(1-x^2)^2 - k_F^4(1-x)^4\right)\Big]$$

$$= \frac{4\pi}{3} k_F^3 \left\{ \frac{1}{3}(1-x)^3 + \frac{1}{2x}\left[\frac{1}{2}(1-x^2)^2 - \frac{1}{2}(1-x^2)(1-x)^2 - \frac{1}{4}(1-x^2)^2 + \frac{1}{4}(1-x)^4\right]\right\}$$

$$= 4\pi k_F^3 (1-x)^2 \left\{ \frac{1}{3}(1-x) + \frac{1}{2x}\underbrace{\left[\frac{1}{4}(1+x)^2 - \frac{1}{2}(1-x^2) + \frac{1}{4}(1-x)^2\right]}_{\frac{1}{4}[(1+x)-(1-x)]^2 \,=\, x^2} \right\}$$

$$= \frac{4\pi}{3} k_F^3 (1-x)^2 \left(1 + \tfrac{1}{2}x\right)$$

(2 points)

Second integral

$$E^{(1)} = -4\pi e^2 \frac{V}{(2\pi)^6} \int \frac{d^3q}{q^2} \frac{4\pi k_F^3}{3} (1-x)^2 (1 + \tfrac{1}{2}x)$$

$$d^3q = 4\pi q^2 dq \quad \text{where } q = 2k_F x$$

$$E^{(1)} = -\frac{(4\pi)^3}{(2\pi)^6} \frac{e^2 k_F^3}{3} \frac{V}{N} N \int_0^1 2k_F\, dx\, (1-x)^2 (1 + \tfrac{1}{2}x)$$

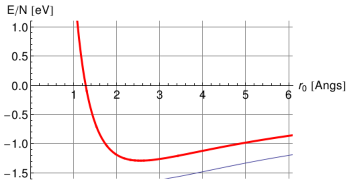
$$\frac{E^{(1)}}{N} = -\frac{1}{\pi^3} \frac{2}{3} \frac{e^2 k_F^4}{n} \cdot \frac{3}{8}$$

; recall $k_F = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_0}$; also $n = \dfrac{1}{\frac{4}{3}\pi r_0^3}$

$$\frac{E^{(1)}}{N} = -\frac{e^2}{2a_0 r_s} \left(\frac{9\pi}{4}\right)^{1/3} \frac{3}{2\pi}$$

$$\therefore \frac{k_F^4}{n} = \frac{1}{r_0}\left(\frac{9\pi}{4}\right)^{1/3} \frac{3\pi^2}{4}$$

(2 points)



Problem 7 We have (eq 3,34)

$$E = E^{(0)} + E^{(1)}$$

where $$E^{(1)} = \frac{-e^2}{2} \frac{4\pi V}{(2\pi)^6} 2 \int \frac{d^3q}{q^2} \int d^3P \, \theta(k_F - |\vec{P} + \tfrac{1}{2}\vec{q}|) \, \theta(k_F - |\vec{P} - \tfrac{1}{2}\vec{q}|)$$

Now replace $\frac{1}{q^2}$ by $\frac{1}{q^2 + (1/d)^2}$ where d is the screening distance.

$$E^{(1)} = -4\pi e^2 \frac{V}{(2\pi)^6} \tfrac{4}{3}\pi k_F^3 \cdot 2k_F \int_0^1 dx \cdot 4\pi \left(1 - \tfrac{3}{2}x + \tfrac{1}{2}x^3\right) \frac{x^2}{x^2 + \mu^2}$$

$$= -\,Ry\, N \frac{0.916}{(r_0/a_B)} \frac{\int_0^1 dx \left(1 - \tfrac{3}{2}x + \tfrac{1}{2}x^3\right) \frac{x^2}{x^2+\mu^2}}{3/8}$$

$\boxed{\mu}$

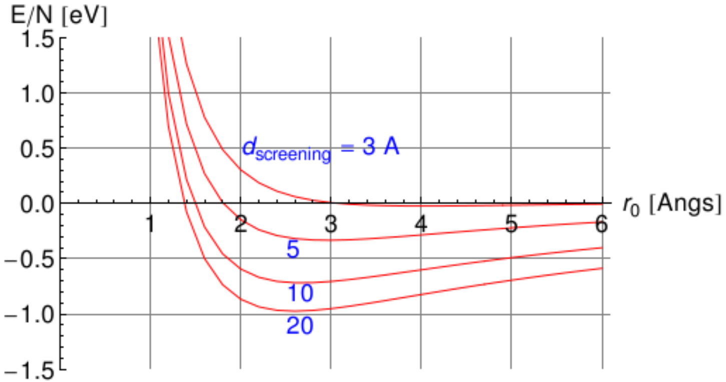
$$\frac{q^2}{q^2 + (1/d)^2} = \frac{4k_F^2 x^2}{4k_F^2 x^2 + \frac{1}{d^2}} = \frac{x^2}{x^2 + (1/2k_F d)^2}$$

$$\mu = \frac{1}{2k_F d}$$

d = screening distance

$$k_F = (3\pi^2 n)^{1/3} = \left(3\pi^2 \frac{N}{\frac{4}{3}\pi r_0^3 N}\right)^{1/3}$$

$$k_F = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_0}$$



Problem 8 (FW #1,2)

(A) The hamiltonian is $\hat{H} = \hat{T} + \hat{V}$

$$\hat{T} = \sum_{rs} T_{rs}\, a_r^\dagger a_s$$

$$\hat{V} = \frac{1}{2}\sum_{rstu} V_{rstu}\, a_r^\dagger a_s^\dagger a_u a_t$$

where

$$T_{rs} = \langle \vec{k}\lambda | \frac{-\hbar^2\nabla^2}{2m} | \vec{q}\mu\rangle$$

$$= \delta_{\lambda\mu} \frac{\hbar^2 k^2}{2m}\delta(\vec{k},\vec{q})$$

and

$$V_{rstu} = \langle \vec{k}\lambda\, \vec{k}'\lambda' | V_0 | \vec{q}\mu\, \vec{q}'\mu'\rangle$$

For the non interacting ground state $|F\rangle$,

$$\langle F | a_r^\dagger a_s | F\rangle = \langle F | a_{k\lambda}^\dagger a_{q\mu} | F\rangle$$

$$= \delta_{\lambda\mu}\,\delta(\vec{k},\vec{q})\,\theta(k_F - k)$$

Therefore

$$\langle F|\hat{T}|F\rangle = \sum_{\vec{k}\lambda}^{k_F} \frac{\hbar^2 k^2}{2m} = 2\sum_{\vec{k}}^{k_F} \frac{\hbar^2 k^2}{2m}$$

That is $E^{(0)}$.

For the non interacting ground state $|F\rangle$,

$$\langle F | a_r^\dagger a_s^\dagger a_u a_t | F\rangle = \delta_{ru}\delta_{st}\langle F | a_r^\dagger a_s^\dagger a_r a_s | F\rangle$$

$$+ \delta_{rt}\delta_{su}\langle F | a_r^\dagger a_s^\dagger a_s a_r | F\rangle$$

$$= \delta_{\lambda\mu'}\,\delta_{\lambda'\mu}\,\delta(\vec{k},\vec{q}')\,\delta(\vec{k}',\vec{q})(-1)\,\theta(k_F-k)\,\theta(k_F-k')$$

$$+ \delta_{\lambda\mu}\,\delta_{\lambda'\mu'}\,\delta(\vec{k},\vec{q})\,\delta(\vec{k}',\vec{q}')(+1)\,\theta(k_F-k)\,\theta(k_F-k')$$

$$\boxed{\begin{aligned} r &= \vec{k}\lambda \\ s &= \vec{k}'\lambda' \\ t &= \vec{q}\mu \\ u &= \vec{q}'\mu' \end{aligned}}$$

Also note: $\vec{k} \neq \vec{k}'$.

Therefore

$$\langle F|\hat{V}|F\rangle = \frac{1}{2}\sum_{\vec{k}\lambda}^{k_F}\sum_{\vec{k}'\lambda'}^{k_F}\Big\{ \langle \vec{k}\lambda\,\vec{k}'\lambda' | V_0 | \vec{k}\lambda\,\vec{k}'\lambda'\rangle - \langle \vec{k}\lambda\,\vec{k}'\lambda' | V_0 | \vec{k}'\lambda'\,\vec{k}\lambda\rangle\Big\}$$

That is $E^{(1)}$. $\boxed{\text{2 points}}$ for part A

(B) For a spin-independent interaction,

$$\langle \vec{k}\lambda\,\vec{k}'\lambda' | V_0 | \vec{k}\lambda\,\vec{k}'\lambda'\rangle = \delta_{\lambda\lambda}\,\delta_{\lambda'\lambda'}\,\langle \vec{k}\vec{k}' | V_0 | \vec{k}\vec{k}'\rangle$$

$$\langle \vec{k}\lambda\,\vec{k}'\lambda' | V_0 | \vec{k}'\lambda'\,\vec{k}\lambda\rangle = \delta_{\lambda\lambda'}\,\delta_{\lambda'\lambda}\,\langle \vec{k}\vec{k}' | V_0 | \vec{k}'\vec{k}\rangle$$

Then

$$\langle F|\hat{V}|F\rangle = \frac{1}{2}\sum_{\vec{k}}^{k_F}\sum_{\vec{k}'}^{k_F}\Big\{ 4\,\langle \vec{k}\vec{k}' | V_0 | \vec{k}\vec{k}'\rangle - 2\,\langle \vec{k}\vec{k}' | V_0 | \vec{k}'\vec{k}\rangle\Big\}$$

For a central interaction $V_0(\vec{x},\vec{y}) = V_0(\vec{\xi})$ where $\vec{\xi} = \vec{x} - \vec{y}$

$$\langle \vec{k}\vec{k}' | V_0 | \vec{k}\vec{k}'\rangle = (1/V)\int V_0(\vec{\xi})\, d^3\xi$$

$$\langle \vec{k}\vec{k}' | V_0 | \vec{k}'\vec{k}\rangle = (1/V)\int V_0(\vec{\xi})\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{\xi}}\, d^3\xi$$

Thus $E^{(1)} = \frac{V}{(2\pi)^3}\int^{k_F} d^3k \, \frac{V}{(2\pi)^3}\int^{k_F} d^3k' \, \frac{1}{V}$

$$\int V_0(\vec{\xi})\, d^3\xi \left[2 - e^{-i(\vec{k}-\vec{k}')\cdot\vec{\xi}} \right]$$

Assume $V_0(\vec{\xi}) < 0$ for all ξ, and $V_0(\vec{\xi}) = V_0(|\vec{\xi}|)$.

Note that $2 - e^{-i(\vec{k}-\vec{k}')\cdot\vec{\xi}} \cancel{\geq} 1$.

Or rather $2 - e^{-i(\vec{k}-\vec{k}')\cdot\vec{\xi}} = 2 - \frac{1}{2}e^{-i(\vec{k}-\vec{k}')\cdot\vec{\xi}} - \frac{1}{2}e^{i(\vec{k}-\vec{k}')\cdot\vec{\xi}}$

$= 2 - \cos\left((\vec{k}-\vec{k}')\cdot\vec{\xi}\right) \geq 1$.

Then $E^{(1)} \leq \frac{V}{(2\pi)^6}\left(\frac{4}{3}\pi k_F^3\right)^2 \underbrace{\int V_0(\vec{\xi})\, d^3\xi}_{\text{(This is negative.)}}$

Recall $N = \frac{2V}{(2\pi)^3}\frac{4}{3}\pi k_F^3$

so $n = \frac{N}{V} = \frac{2}{(2\pi)^3}\left(\frac{4}{3}\pi k_F^3\right)$; or $k_F = (3\pi^2 n)^{1/3}$.

$$\frac{E^{(1)}}{N} \leq \frac{V}{N}\left(\frac{N}{2V}\right)^2 \int V_0(\vec{\xi})\, d^3\xi = \frac{n}{4}\int V_0(\vec{\xi})\, d^3\xi$$

Recall $E^{(0)} = 2\sum_{\vec{k}}^{k_F} \frac{\hbar^2 k^2}{2m}$

$= 2(4\pi)\frac{\hbar^2}{2m}\frac{k_F^5}{5}\frac{V}{(2\pi)^3}$

$$\frac{E^{(0)}}{N} = \frac{V}{N}\frac{3}{5}k_F^2 \underbrace{\frac{2\left(\frac{4}{3}\pi k_F^3\right)}{(2\pi)^3}}_{n} = \frac{3}{5}\left(3\pi^2 n\right)^{2/3}$$

Thus

$$\frac{E^{(0)} + E^{(1)}}{N} = c_1 n^{2/3} - c_2 n$$

when c_1 and c_2 are positive.

The minimum occurs at $E/N = -\infty$ when $n \to \infty$. Thus the system will collapse to infinite density.

$\boxed{\text{4 points}}$ for part B