# *Chapter 1 : SECOND QUANTIZATION*

***Readings*** and ***lectures*** ...

You are going to read the chapter at home.

I will go over some of the calculations in class, but not all of them.

The purpose of the lectures is to understand the results, but not always to derive them in detail.

You should read the material before the lecture so that you will know the notations.

## 1. THE SCHROEDINGER EQUATION IN FIRST AND SECOND QUANTIZATION

### Many body theory in first quantization

Start with

$$H = \sum^{N}_{k=1} T(x_k) + ½ \sum'^{N}_{k,l=1} V(x_k,x_l) \; ;$$

then find iħ (∂f/∂t) where f is the *amplitude for occupation numbers*

$$f_N ( n_1 \, n_2 \ldots n_i \ldots n_\infty ) \, .$$

Notation

$\sum'$ : here the prime means $l \neq k$

## The first quantized theory

$$H = \sum_{k=1}^{N} T(x_k) + \frac{1}{2}\sum_{k,l=1}^{N}{}' \; V(x_k, x_l)$$

$T(x_k) = -\frac{\hbar^2}{2m}\nabla_k^2$ ; $T(x_k)$ is a "one-body operator"

$V(x_k, x_l)$ = interaction potential; a "two-body operator"

Notations

- Index k labels one of the N particles; the set of k values is $\{k\} = \{1\,2\,3 \ldots k \ldots N\}$
- $x_k$ is a complete set of coordinates for particle k. For example, for an electron

$$x_k = (\underbrace{x_k, y_k, z_k}_{\text{Cartesian coordinates; } \in (0,L)}, \; \xi_k) \longleftarrow \text{spin coordinate; } \in(-\tfrac{1}{2}, \tfrac{1}{2})$$

To Solve: $i\hbar\, \partial\Psi / \partial t = H\, \Psi$

where $\Psi = \Psi(x_1\; x_2\; x_3 \ldots x_k \ldots x_N)$

## Introduce a complete set of time-independent single-particle wave functions

Single-particle states

For a single particle with coordinates $x$ we have $\psi_E(x)$ where $E$ = a set of quantum numbers for the particle.

For example, consider electrons in a box ($V = L^3$) with periodic boundary conditions

$$E = (p_x, p_y, p_z, s_z)$$

$$\psi_E(x) = \frac{1}{\sqrt{V}} e^{i\vec{p}\cdot\vec{x}/\hbar}\, u_{s_z}$$

The periodic B.C.'s imply

$p_x = \frac{2\pi\hbar}{L} n_x$ where $n_x$ is an integer

$p_y = \frac{2\pi\hbar}{L} n_y$ and $p_z = \frac{2\pi\hbar}{L} n_z$

Also, $u_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}$ and $u_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$

The single-particle states must be complete.

# Completeness: We can expand the N-particle wave function as a product of single-particle wave functions __

Since the single-particle states are complete, we can expand $\Psi(x_1, x_2 \dots x_k \dots x_N)$ as

$$\sum_{E_1'} \sum_{E_2'} \dots \sum_{E_N'} C(E_1' E_2' \dots E_N' ; t) \prod_{k=1}^{N} \psi_{E_k'}(x_k)$$

Or, in a shorter notation

$$\Psi(\{x\};t) = \sum_{\{E'\}} C(\{E'\};t) \prod_{k=1}^{N} \psi_{E_k'}(x_k)$$

By the <u>orthonormality</u> of s.p. states

$$C(\{E\};t) = \int \prod_{k=1}^{N} \psi^{\dagger}_{E_k}(x_k)\, \Psi(\{x\};t) \prod_k (d\tau)_k$$

$$(d\tau)_k = d^3x_k \sum_{\xi_k}$$

The meaning of the expansion is that $|C(\{E\};t)|^2$ = the probability that the particles have the quantum numbers, $\{E_1 E_2 \dots E_k \dots E_N\}$.

## Normalization relations

$$\int \Psi^{\dagger} \Psi \prod_{k=1}^{N} (d\tau)_k = 1$$ ; which implies

$$\sum_{\{E\}} |C(\{E\}, t)|^2 = 1.$$

## The Schroedinger equation for $C(\{E\};t)$

$$i\hbar \frac{\partial}{\partial t} C(\{E\},t) = \int \prod_{\ell=1}^{N} \psi^{\dagger}_{E_\ell}(x_\ell)\, \underbrace{i\hbar \frac{\partial \Psi}{\partial t}}_{} \prod_{k=1}^{N} (d\tau)_k$$

$$H\Psi = T\Psi + V\Psi$$

$\sum_W |W\rangle\langle W| = 1$

$|WW'\rangle\langle WW'|$

→ eq.(1.4)

$$= \sum_{k=1}^{N} \sum_{W} \langle E_k | T | W \rangle\, C(E_1 \dots E_{k-1}\, W\, E_{k+1} \dots E_N ; t)$$

← $E_K$ replaced by $W$

$$+ \frac{1}{2} \sum_{k,\ell=1}^{N}{}' \sum_{W} \sum_{W'} \langle E_k E_\ell | V | W W' \rangle\, C(E_1 \dots E_{k-1}\, W\, E_{k+1} \dots E_{\ell-1}\, W'\, E_{\ell+1} \dots E_N)$$

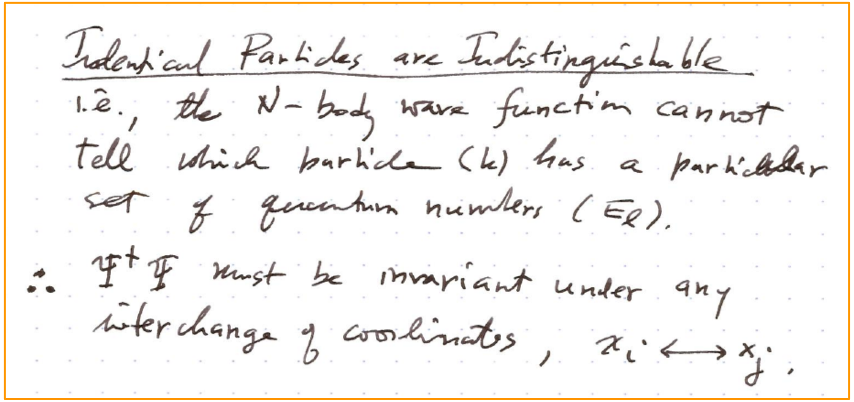
where

$$\langle E_k | T | W \rangle = \int \psi^{\dagger}_{E_k}(x)\, T(x)\, \psi_W(x)\, (d\tau)$$

and

$$\langle E_k E_\ell | V | W W' \rangle = \iint \psi^{\dagger}_{E_k}(x)\, \psi^{\dagger}_{E_\ell}(x')\, V(x,x')\, \psi_W(x)\, \psi_{W'}(x')\, (d\tau)(d\tau')$$

So far, this is simple. Now comes the hard part: the N particles are ***identical particles.***



Identical Particles are Indistinguishable

i.e., the N-body wave function cannot tell which particle (k) has a particular set of quantum numbers ($E_l$).

∴ $\Psi^\dagger \Psi$ must be invariant under any interchange of coordinates, $x_i \longleftrightarrow x_j$.

## 1a. Bosons .

$\Psi(x_1 \ldots x_N ; t)$ must be symmetric with respect to interchange of any two coordinates; i.e.,

$$\Psi( \ldots x_k \ldots x_l \ldots ; t ) = + \Psi( \ldots x_l \ldots x_k \ldots ; t )$$

Then also,

$$C( \ldots E_k \ldots E_l \ldots ; t ) = + C( \ldots E_l \ldots E_k \ldots ; t )$$

*for any case of k and l*

## The space of occupation numbers

In general, $C(E_1 E_2 \ldots E_N ; t)$ could depend on 3N quantum numbers (ignoring spin; set $s=0$).

$$\{ \vec{p}_1\ \vec{p}_2\ \vec{p}_3 \ldots \vec{p}_k \ldots \vec{p}_N \}$$

But the function must be symmetric w.r.t. interchanges. Therefore $C(E_1 E_2 \ldots E_N ; t)$ might depend on as few as 3 quantum numbers

$$\{ \vec{p}_1\ \underset{\underset{\vec{p}_2}{\shortparallel}}{\vec{p}_1}\ \underset{\underset{\vec{p}_3}{\shortparallel}}{\vec{p}_1} \ldots \underset{\underset{\vec{p}_k}{\shortparallel}}{\vec{p}_1} \ldots \underset{\underset{\vec{p}_N}{\shortparallel}}{\vec{p}_1} \}$$

In fact, $C$ depends only on the set of "occupation numbers".

**Define occupation number $n_i$ = the number of particles with quantum numbers $E_i$.**

Note that $n_i$ could be any integer in the range from 0 to N. (for bosons!)

Restriction: $\sum_{i=1}^{\infty} n_i = N$

Imagine that the s.p. states can be listed in some order $\{ E_1\ E_2\ E_3 \ldots E_\infty \}$.

"standard order"

The set of occupation numbers is $\{ n_1\ n_2\ n_3 \ldots n_\infty \}$ and $\sum_{i=1}^{\infty} n_i = N$.

(Notation: k labels a particle; i labels a state)
$\{k\} = \{1\,2\,3 \ldots N\}$ $\{i\} = \{1\,2\,3 \ldots \infty\}$

The basis states of the system are $| n_1\ n_2\ n_3 \ldots n_\infty \rangle$.

Most $n_i$'s are 0.

So,

$$C(E'_1\ E'_2 \ldots E'_k \ldots E'_N ; t) = \tilde{C}( n_1\ n_2 \ldots n_\infty ; t)$$

where $n_i = \sum_{k=1}^{N} \delta_{Kr.}(E_i, E'_k)$

$C(\{E\};t)$ is totally symmetric in $\{E\}$

$\tilde{C}(\{n\};t)$ has no symmetry property; only the restriction $\sum_i n_i = N$.

## Meaning of $\tilde{C}(\{n\};t)$

$|\tilde{C}(\{n\};t)|^2$ is the probability that the list of occupation numbers is

$\{n\} = \{n_1\, n_2 \ldots n_i \ldots \infty\}$.

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Previously we had sums or products over particle labels; i.e., $\sum_{k=1}^{N}$ or $\prod_{k=1}^{N}$.

We will replace these by sums or products over states ( [redacted] ; i.e. occupation numbers ;

i.e. $\sum_{n_i=0}^{N}$ or $\prod_{n_i=0}^{N}$ .

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## The normalization of the wave function

$$\int \Psi^{+} \Psi \prod_{k=1}^{N} (d\tau)_k = 1 \quad \text{or} \quad \sum_{\{E\}} |C(\{E\};t)|^2 = 1$$

Now

$$\sum_{\{E\}} |\tilde{C}(\{n\};t)|^2 = 1 \qquad \boxed{\tilde{C}(\{n\}) = C(\{E\})}$$

$$\hookrightarrow \; = \sum_{\{n\}}{}' \frac{N!}{n_1!\, n_2!\, n_3! \ldots n_\infty!} |\tilde{C}(\{n\},t)|^2$$

where $\Sigma'$ means sum restricted to $\sum_i n_i = N$.

$$\frac{N!}{\prod_{i=1}^{\infty} n_i!}$$

= the number of permutations of $\{E_1\, E_2 \ldots E_N\}$ such that the list of quantum numbers is $\{n_1\, n_2\, n_3 \ldots n_\infty\}$

$\boxed{0! = 1}$

Now define the occupation number probability amplitude

$$f_N(\{n\};t) = \sqrt{\frac{N!}{\prod_{i=1}^{\infty} n_i!}}\, \tilde{C}(\{n\};t)$$

Then $\sum_{\{n\}}{}' |f_N(\{n\};t)|^2 = 1$.

and express the coordinate space wavefunction $\Psi(x_1\, x_2 \ldots x_k \ldots x_N)$ in terms of $f_N(\{n\};t)$

$$\Psi = \sum_{\{n\}}{}' f_N(\{n\};t)\, \Phi_{\{n\}}(\{x\})$$

where

$$\Phi_{\{n\}}(\{x\}) = \sqrt{\frac{\prod_i n_i!}{N!}} \sum_{\{E_1 \ldots E_N\}}{}'' \psi_{E_1}(x_1)\, \psi_{E_2}(x_2) \ldots \psi_{E_N}(x_N)$$

$''$ means the occupation number of $\cancel{E}_k = n_k \quad \forall k$

$$\boxed{\Phi_{\{n\}}(\{x\}) = \left(\frac{\prod_i n_i!}{N!}\right)^{1/2} \sum_{\{E_1 \dots E_N\}}'' \prod_{k=1}^{N} \psi_{E_k}(x_k)}$$

These are just the symmetric, orthonormal basis states for N identical bosons.

Example: $N=2$

The basis states are

$$\sqrt{\frac{2!}{2!}}\,\psi_{E_i}(x_1)\,\psi_{E_i}(x_2)$$

(both particles in the same s.p. state $E_i$)

and

$$\sqrt{\frac{1!\,1!}{2!}}\left[\psi_{E_i}(x_1)\,\psi_{E_j}(x_2) + \psi_{E_j}(x_1)\,\psi_{E_i}(x_2)\right]$$

(one in state $E_i$, the other in state $E_j$; $j \neq i$)

note that the states are orthonormal — i.e. orthogonal if the lists of occupied states are different, and normalized

$$\int \Phi^{+}\Phi\,(d\tau_1)(d\tau_2) = 1.$$

Another notation: sum over permutations

$$\Phi_{\{n\}}(\{x\}) = \sqrt{\frac{\prod_i n_i!}{N!}} \sum_{P} \psi_{E_1'}(x_1)\dots\psi_{E_k'}(x_k)\cdots\psi_{E_N'}(x_N)$$

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The Schroedinger equation for $f_N(\{n\};t)$

$$i\hbar\frac{\partial}{\partial t} f_N(\{n\};t) = \left(\frac{N!}{\prod_i n_i!}\right)^{1/2} \underbrace{i\hbar\frac{\partial}{\partial t} C(\{E\};t)}_{\text{See earlier}}$$

The result is derived in F.W.;
The final equation is (1,25)

Eq. 1.25

$$= \sum_i \langle i|T|i\rangle\, n_i\, f_N(\{n\};t)$$

$$+ \sum_{i,j}{}' \langle i|T|j\rangle \sqrt{n_i}\cdot\sqrt{n_j+1}\; f_N(\{n\}+\Delta_{ij};t)$$

$$+ \sum_{ijkl}{}' \langle ij|V|kl\rangle\, \tfrac{1}{2}\sqrt{n_i}\sqrt{n_j}\sqrt{n_k+1}\sqrt{n_l+1}\; f_N(\{n\}+\Delta_{ik}+\Delta_{jl};t)$$

(ijkl all different)

$$+ \sum_{i\,kl}{}' \langle ii|V|kl\rangle\, \tfrac{1}{2}\sqrt{n_i}\sqrt{n_i}\cdot\sqrt{n_k+1}\sqrt{n_l+1}\; f_N(\{n\}+\Delta_{ik}+\Delta_{il};t)$$

(i=j; k l different)

+ similar terms

where $\Delta_{ij} = \{0\,0 \dots 0\,{-1}\,0 \dots 0\,{+1}\,0 \dots 0\,0\}$

↑ site i ↑ site j

"FOURTEEN "Similar terms"" ijkl all different (1)

\+ 2 the same, the others different (3+2+1=6)

\+ 2 the same, the other two are the same (but different from the first two) (3)

\+ 3 the same, final one different (3)

\+ all the same (1)

(Lecture Jan 14)

## Summary so far -- the quantum many-particle problem in first quantized form ...

- understand that the symmetry of the wave function (for bosons) implies that the basis states depend only on the list of occupation numbers;

$$\Phi_{\{n\}}(\{x\}) = \sqrt{\frac{\prod_i n_i!}{N!}} \sum_P \psi_{E_1'}(x_1) \dots \psi_{E_k'}(x_k) \dots \psi_{E_N'}(x_N)$$

basis states;

$$\Psi(\{x\},t) = {\sum_{\{n\}}}' f_N(\{n\},t)\, \Phi_{\{n\}}(\{x\})$$

- the Schroedinger equation for the occupation number amplitude (eq. 1.25);

$$i\hbar \frac{\partial}{\partial t} f_N(\{n\},t) = \sum_i \langle i|T|i\rangle\, n_i\, f_N(\{n\},t)$$

$$+ \sum_{ij}^{i\neq j} \langle i|T|j\rangle \sqrt{n_i}\sqrt{n_j+1}\; f_N(\{n\}+\Delta_{ij},t) \qquad \Delta_{ij} = \{0\dots\overset{(i)}{-1}\dots\overset{(j)}{+1}\dots 0\}$$

$$+ \sum_{ijkl}^{\text{ijkl all different}} \langle ij|V|kl\rangle \tfrac{1}{2}\sqrt{n_i}\sqrt{n_j}\sqrt{n_k+1}\sqrt{n_l+1}\; f_N(\{n\}+\Delta_{ik}+\Delta_{jl},t)$$

$$+ \sum_{ikl}^{\text{ikl all different}} \langle ii|V|kl\rangle \tfrac{1}{2}\sqrt{n_i}\sqrt{n_i-1}\sqrt{n_k+1}\sqrt{n_l+1}\; f_N(\{n\}+\Delta_{ik}+\Delta_{il},t)$$

$$+ \text{ etc.} \qquad (1.25)$$

# 1b. The many-particle Hilbert space (*a.k. a.* Fock space); creation and annihilation operators; SECOND QUANTIZATION

- Basis states

$$|n_1 n_2 n_3 \dots n_\infty\rangle \quad \text{or} \quad |\{n\}\rangle$$

where $\sum_{i=1}^{\infty} n_i = N$.

- orthonormality

$$\langle \{n'\} | \{n\} \rangle = \delta(n_1', n_1)\, \delta(n_2', n_2) \dots \delta(n_\infty', n_\infty)$$

$$= \prod_{i=1}^{\infty} \delta(n_i', n_i) \quad \text{(Kronecker delta)}$$

- Completeness

$$\sum_{\{n\}}{}' \; |\{n\}\rangle\langle\{n\}| = \mathbb{1}$$

- Annihilation and creation operators

$$b_i \quad \text{and} \quad b_i^\dagger$$

are defined by these commutation relations

$$[b_i, b_j] = 0 \quad \text{and} \quad [b_i^\dagger, b_j^\dagger] = 0$$

$$[b_i, b_j^\dagger] = \delta_{ij}$$

bosons

You should remember these commutation relations, from the raising and lowering operators of the harmonic oscillator.

Given

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad \text{where } [p, x] = -i\hbar$$

let

$$a = \frac{p}{\sqrt{2m\hbar\omega}} - i\sqrt{\frac{m\omega}{2\hbar}}\, x$$

$$a^\dagger = \frac{p}{\sqrt{2m\hbar\omega}} + i\sqrt{\frac{m\omega}{2\hbar}}\, x$$

Then $[a, a^\dagger] = 0 + 0 + \frac{2}{\sqrt{2m\hbar\omega}} (+i) \sqrt{\frac{m\omega}{2\hbar}} \underbrace{[p, x]}_{-i\hbar}$

$$[a, a^\dagger] = 1$$

and of course $[a, a] = 0$.

And, $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$

The eigenvalues of $a^\dagger a$ are integers $\geq 0$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

where $a^\dagger a|n\rangle = n\,|n\rangle$.

Important $b_i$ and $b_i^+$ have nothing to do with harmonic oscillator hamiltonian; but the same commutation relations

$\{b_i; b_i) = 0$ ; $[b_i; b_i^+] = 1$ ; $n_i = b_i^+ b_i$.

## Creation and annihilation operators

⊕ $b_i^+ b_i$ is Hermitian, so its eigenvalues are real.

⊕ $\langle 0| b_i^+ b_i |0\rangle \geq 0$ so the lowest eigenvalue is 0,

⊕ $b_i^+ |0\rangle = |1\rangle$

Proof $b_i^+ b_i \; b_i^+ |0\rangle = b_i^+ (\underbrace{b_i b_i^+ - b_i^+ b_i}_{1} + \underbrace{b_i^+ b_i}_{0}) |0\rangle$

$= 1 \; b_i^+ |0\rangle$ QED

⊕ $b_i^+ |n\rangle = \sqrt{n+1} \; |n+1\rangle$

Proof(i) $b_i^+ b_i \; b_i^+ |n\rangle = b_i^+ [\underbrace{[b_i, b_i^+]}_{1} + \underbrace{b_i^+ b_i}_{n}] \; |n\rangle$

$= (n+1) \; b_i^+ |n\rangle$

and(ii) $\langle n| b_i b_i^+ |n\rangle = \langle n| 1 + b_i^+ b_i |n\rangle = n+1$.

QED

⊕ $b_i |n\rangle = \sqrt{n} \; |n-1\rangle$

proof(i) $b_i^+ b_i \; b_i |n\rangle = (\underbrace{b_i^+ b_i - b_i b_i^+}_{-1} + b_i b_i^+) b_i |n\rangle$

$= - b_i |n\rangle + b_i (\underbrace{b_i^+ b_i}_{n}) |n\rangle = (n-1) \; b_i |n\rangle$

and(ii) $\langle n| b_i^+ b_i |n\rangle = n$ QED

⊕ $b_i^+ b_i$ = the number operator

$b_i^+$ = the creation operator

$b_i$ = the annihilation operator.

Now, write the Hamiltonian in terms of $b_i$ and $b_j^+$ operators.

$$H = \sum_{ij} b_i^+ < i\,|T|\,j > b_j + ½ \sum_{ij} b_i^+ \, b_j^+ < ij\,|V|\,kl > b_l \, b_k$$

$$\hat{H} = \sum_{ij} b_i^+ \langle i|T|j\rangle b_j + \frac{1}{2}\sum_{ijkl} b_i^+ b_j^+ \langle ij|V|kl\rangle b_l b_k$$

where

$$\langle i|T|j\rangle = \int d^3x \; \psi_i^+(x)\, T(x)\, \psi_j(x)$$

and

$$\langle ij|V|kl\rangle = \int d^3x\, d^3y \; \psi_i^*(x)\, \psi_j^+(y)\, V(x,y)\, \psi_k(x)\, \psi_l(y)$$

Simplify the notation:

$\psi_i(x)$ means $\psi_{Ei}(x)$;

i.e., the single particle wave function with quantum numbers $E_i$ .

# Proof

$$|\psi,t\rangle = \sum_{\{n\}} f_N(\{n\}) \prod_{i=1}^{\infty} \overbrace{\frac{(b_i^+)^{n_i}}{\sqrt{n_i!}}|0\rangle}^{|n_i\rangle \text{ normalized}}$$

$$i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = \sum_{\{n\}} f_N(\{n\})\Big\{\sum_{ij}\langle i|T|j\rangle b_i^+ b_j |\{n\}\rangle$$

$$+1/2\sum_{ijkl}\langle ij|V|kl\rangle b_i^+ b_j^+ b_k b_l |\{n\}\rangle\Big\}$$

$$T\text{part} = \sum_{\{n\}} f_N(\{n\})\Big[\sum_i \langle i|T|i\rangle n_i |\{n\}\rangle + \sum_{i\neq j}\langle i|T|j\rangle\sqrt{n_i+1}\sqrt{n_j}\,|\underbrace{\{n\}-\Delta_{ij}}_{\equiv\{n'\}}\rangle$$

$$n_i' = n_i+1$$
$$n_j' = n_j-1$$

$$= \sum_{\{n\}} f_N(\{n\}) \sum_i \langle i|T|i\rangle n_i |\{n\}\rangle + \sum_{\{n'\}} f_N(\{n'\}+\Delta_{ij}) \sum_{i\neq j}\langle i|T|j\rangle\sqrt{n_i'}\sqrt{n_j'+1}\,|\{n'\}\rangle$$

$$V_{part} = \sum_{\{n\}} f_N(\{n\})\, 1/2 \underset{\text{(all different)}}{\sum_{ijkl}} \langle ij|V|kl\rangle\sqrt{n_i+1}\sqrt{n_j+1}\sqrt{n_k}\sqrt{n_l}\,|\{n\}-\Delta_{ij,kl}\rangle$$

+etc.

$$\Delta_{ij} = \{0\ldots \underset{(i)}{-1} \ldots \underset{(j)}{+1} \ldots 0\}$$

The final result is equivalent to Eq. (1.25).
Q.E.D.

## 1c. Fermions

Start again with the first quantized N-body Hamiltonian ,

$$H = \sum_{k=1}^{N} T(x_k) + \frac{1}{2} \sum_{k,l=1}^{N}{}' V(x_k, x_l)$$

(T and V would be matrices w.r.t. spin space.)

Now add the ***antisymmetry*** of the of the wave function $\Psi(x_1\ x_2 \ldots x_k \ldots x_N ; t)$ ,

$$\Psi( \ldots x_k \ldots x_k \ldots \ ; t)$$
$$= - \Psi( \ldots x_l \ldots x_k \ldots \ ; t)$$

*for any case of k and k , for fermions.*

**Exchange includes spin indices, suppressed here.

Fermions are easier than bosons because the occupation numbers are more limited:

for any single-particle state i,
$n_i$ can only be 0 or 1.

That's the *Pauli exclusion principle*.

It is a consequence of the antisymmetry of the wave function.

$$\ldots \psi_E (x_k) \ldots \psi_{E'} (x_l) \ldots$$

must be equal to

$$- \ldots \psi_E (x_l) \ldots \psi_{E'} (x_k) \ldots$$

If E' = E then the N-body wave function could only be 0; not possible.

## Annihilation and creation operators for fermions.

Use notation $c_i$ and $c_i^\dagger$ for fermions.

FOR FERMIONS, THE DEFINING COMMUTATION RELATIONS ARE *ANTI* COMMUTATION RELATIONS.

$$\{ c_i , c_j \} = c_i c_j + c_j c_i = 0$$

$$\{ c_i^\dagger , c_j^\dagger \} = c_i^\dagger c_j^\dagger + c_j^\dagger c_i^\dagger = 0$$

$$\{ c_i , c_j^\dagger \} = c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij} \quad .$$

Another notation for the anticommutator:

$\{A,B\} = [A,B]_+$

The interpretation of fermionic operators ($c_i$ and $c_i^\dagger$ ) is the same as for bosonic operators ($b_i$ and $b_i^\dagger$ ):

$c_i$ = annihilation operator; annihilates a particle in the state $E_i$ ;
$c_i^\dagger$ = creation operator; creates a particle in the state $E_i$ ;
$c_i^\dagger c_i$ = occupation number operator for the state $E_i$ .

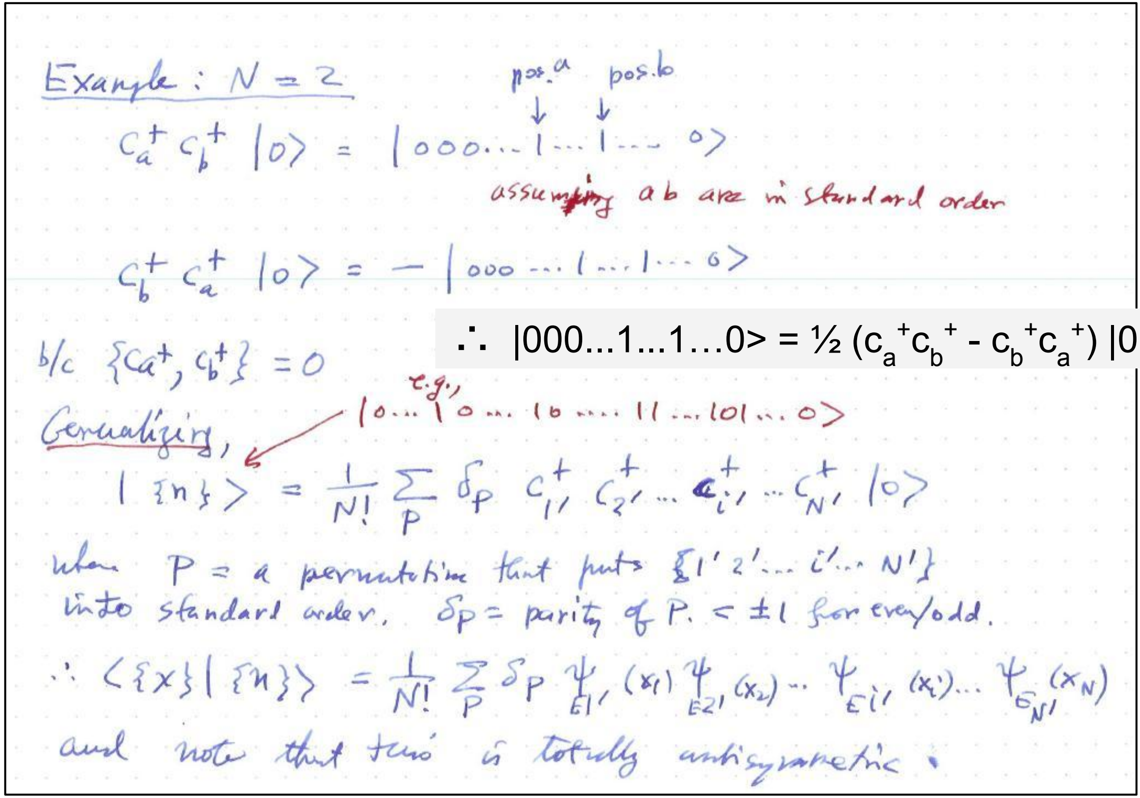
*Note that this automatically satisfies the Pauli exclusion principle:*
A state with two particles would be

$$c_i^\dagger c_j^\dagger |0\rangle ;$$

but if i = j then we would have $c_i^\dagger c_i^\dagger$ ; but this is 0 by the second A.C. relation. So two particles cannot occupy the same s.p. state.

# Antisymmetry of the wave function

**Theorem.**
The ***anti*** commutation relations of the creation and annihilation operators imply that the N-body wave function will be antisymmetric.



Example : N = 2

pos. a pos. b

$$c_a^+ c_b^+ |0\rangle = |000 \ldots 1 \ldots 1 \ldots 0\rangle$$

assuming a b are in standard order

$$c_b^+ c_a^+ |0\rangle = - |000 \ldots 1 \ldots 1 \ldots 0\rangle$$

b/c $\{c_a^+, c_b^+\} = 0$

∴ $|000 \ldots 1 \ldots 1 \ldots 0\rangle = \frac{1}{2} (c_a^+ c_b^+ - c_b^+ c_a^+) |0\rangle$

Generalizing,

e.g., $|0 \ldots 1 0 \ldots 1 0 \ldots 1 1 \ldots 1 0 1 \ldots 0\rangle$

$$|\{n\}\rangle = \frac{1}{N!} \sum_P \delta_P \, c_{1'}^+ c_{2'}^+ \ldots c_{i'}^+ \ldots c_{N'}^+ |0\rangle$$

when P = a permutation that puts $\{1' 2' \ldots i' \ldots N'\}$ into standard order. $\delta_P$ = parity of P. ← ±1 for even/odd.

$$\therefore \langle \{x\} | \{n\} \rangle = \frac{1}{N!} \sum_P \delta_P \, \psi_{\epsilon 1'}(x_1) \psi_{\epsilon 2'}(x_2) \ldots \psi_{\epsilon i'}(x_i) \ldots \psi_{\epsilon N'}(x_N)$$

and note that this is totally antisymmetric.