Chapter 3 : Green's functions and field theory (fermions)

Review

$$H = \int \psi^{\dagger}(x)\, T(x)\, \psi(x)\, d^3x$$
$$+\tfrac{1}{2} \iint \psi^{\dagger}(x)\, \psi^{\dagger}(x')\, V(x,x')\, \psi(x')\, \psi(x)\, d^3x d^3x'$$

$$\{\psi(x)\, ,\, \psi(x')\} = 0$$

$$\{\psi(x)\, ,\, \psi^{\dagger}(x')\} = \delta^3(x-x')$$

(spin indices are suppressed)

6. PICTURES

The predictions of a quantum theory depend entirely on matrix elements;

$$\langle\, \alpha \,|\, Q \,|\, \beta\, \rangle = Q_{\alpha\beta}(t)\ .$$

Now which parts of the theory (i.e., states or operators) depend on time?

Schroedinger picture: the states depend on time and the operators do not depend on time.

Heisenberg picture: the operators depend on time and the states do not depend on time.

Interaction picture: both states and operators depend on time.

The *matrix elements* , and hence *predictions*, must be equal in all three pictures. For example,

$$\langle \alpha_S(t) \,|\, Q_S \,|\, \beta_S\, (t) \rangle = \langle \alpha_H \,|\, Q_H\, (t) \,|\, \beta_H \rangle\ .$$

6a. The Schroedinger picture

(the most familiar)

$$i\hbar \frac{\partial}{\partial t} |\Psi_S(t)\rangle = H |\Psi_S(t)\rangle$$

The formal solution of this equation is

$$|\Psi_S(t)\rangle = e^{-iH(t-t_0)/\hbar} |\Psi_S(t_0)\rangle$$

H is Hermitian ($H^{\dagger} = H$)

So $e^{-iH(t-t_0)/\hbar}$ is unitary ($U^{\dagger}U = 1$).

Observables are Hermitian operators.

Consider $\left[\frac{-i(t-t_0)}{\hbar}\right]^n \frac{H^n}{n!}$

$$\partial/\partial t(\,''\,) = n\left(\frac{-i}{\hbar}\right)\left[\frac{-i(t-t_0)}{\hbar}\right]^{n-1} \frac{H^n}{n!}$$

$$= \frac{-i}{\hbar} H \left[\frac{-i(t-t_0)}{\hbar}\right]^{n-1} \frac{H^{n-1}}{(n-1)!}$$

6b. The interaction picture

(useful for perturbation theory)

Suppose $H = H_0 + H_1$

where H_0 is solvable (usually H_0 = free particle) and H_1 is an interaction (small if the particles are far apart, so amenable to perturbation theory).

Define this unitary transformation

$$|\Psi_I(t)\rangle = e^{iH_0 t/\hbar}\,|\Psi_S(t)\rangle$$

($\Psi_S \rightarrow \Psi_I$ by the action of $e^{iH_0 t/\hbar}$)

Derive the time evolution of the I.P. state

$$i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = \left\{ i\hbar\, e^{iH_0 t/\hbar}\,\frac{iH_0}{\hbar} + e^{iH_0 t/\hbar} H \right\} |\Psi_S(t)\rangle$$

$$= e^{iH_0 t/\hbar}(H - H_0)\, e^{-iH_0 t/\hbar}\, e^{iH_0 t/\hbar}|\Psi_S(t)\rangle$$

$$= H_1(t)\,|\Psi_I(t)\rangle$$

where $H_1(t) = e^{iH_0 t/\hbar} H_1 e^{-iH_0 t/\hbar}$

The time evolution of the state is

$$i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = H_1(t)\,|\Psi_I(t)\rangle\,;$$

and the time evolution of an operator is

$$Q_I(t) = e^{iH_0 t/\hbar}\, Q_S\, e^{-iH_0 t/\hbar}.$$

Or,

$$\frac{\partial}{\partial t}Q_I(t) = e^{iH_0 t/\hbar}\,\frac{iH_0}{\hbar}\,Q_S\, e^{-iH_0 t/\hbar} + e^{iH_0 t/\hbar}\, Q_S\left(-\frac{iH_0}{\hbar}\right) e^{-iH_0 t/\hbar}$$

$$= \frac{i}{\hbar}\left[H_0,\, Q_I(t)\right]$$

Note: H_0 commutes with $e^{\pm iH_0 t/\hbar}$

$Q_I(0) = Q_S$

Matrix elements are equal:

$$\langle \Psi_I(t)|\, Q_I(t)\, |\Psi_I(t)\rangle$$

$$= \underbrace{\langle \Psi_S(t)|\, e^{-iH_0 t/\hbar}}\; \overbrace{e^{iH_0 t/\hbar}\, Q_S\, e^{-iH_0 t/\hbar}}\; \underbrace{e^{iH_0 t/\hbar}|\Psi_S(t)\rangle}$$

$$= \langle \Psi_S(t)|\, Q_S\, |\Psi_S(t)\rangle \; \checkmark$$

6c. The Heisenberg picture
(important for proving general theorems)

Consider this unitary transformation

$$|\Psi_H\rangle = e^{iHt/\hbar}\,|\Psi_s(t)\rangle.$$

Note that $|\Psi_H\rangle$ does not depend on t;

$$i\hbar\frac{\partial}{\partial t}|\Psi_H\rangle = \left\{ i\hbar\, e^{iHt/\hbar}\left(\frac{iH}{\hbar}\right) + e^{iHt/\hbar} H \right\}|\Psi_s(t)\rangle$$

$$= 0$$

How do the operators depend on time?

We must have

$$\langle \Psi_H | Q_H(t) | \Psi_H\rangle = \langle \Psi_s(t) | Q_s | \Psi_s(t)\rangle$$

$$= \langle \Psi_s(t) | \underbrace{e^{-iHt/\hbar} Q_H(t) e^{iHt/\hbar}}_{\text{So this must be } Q_s.} | \Psi_s(t)\rangle$$

Thus

$$Q_H(t) = e^{iHt/\hbar}\, Q_s\, e^{-iHt/\hbar}.$$

Or,

$$\frac{\partial}{\partial t} Q_H(t) = \frac{i}{\hbar}[H, Q_H]$$

Perturbation theory and the interaction picture ...

Assume $H = H_0 + H_1$,

where H_0 is solvable and H_1 is a set of interactions, possibly having small effects.

{Usually H_0 is a single particle operator; and H_1 is a two-particle operator describing the interactions between particles.}

How can we calculate the effects of H_1 ?

Let the single-particle states be eigenstates of $T(x)$,

$$-\frac{\hbar^2\nabla^2}{2m}\psi(x) = \epsilon\,\psi(x)$$

$$\Rightarrow \psi(x) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \underbrace{\times \text{ spin state}}$$

Suppress this but keep it in mind.

where $\epsilon = \hbar^2 k^2/2m$.

Then

$$\hat{H}_0 = \sum_{k,\ell} \langle k|T|\ell\rangle\, c_k^\dagger c_\ell$$ in second quantized form

where

$$\langle k|T|\ell\rangle = \frac{1}{V}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \frac{\hbar^2\ell^2}{2m} e^{i\vec{\ell}\cdot\vec{x}}$$

$$= \frac{\hbar^2 k^2}{2m}\,\delta(\vec{k},\vec{\ell})$$ (Kronecker delta)

$$\hat{H}_0 = \sum_k \hbar\omega_k\, c_k^\dagger c_k$$ in the Schroedinger picture.

In the interaction picture,

$$c_k(t) = e^{iH_0 t/\hbar}\, c_{ks}\, e^{-iH_0 t/\hbar}$$

$$\frac{\partial c_k}{\partial t} = e^{iH_0 t/\hbar} \underbrace{\left\{ \frac{i}{\hbar} H_0 c_{ks} - \frac{i}{\hbar} c_{ks} H_0 \right\}}_{\frac{i}{\hbar}\hbar\omega_m [c_m^\dagger c_m, c_k] \ \left(\sum_m\right)} e^{-iH_0 t/\hbar}$$

$$= i\omega_m \left\{ c_m^\dagger c_m c_k - c_k c_m^\dagger c_m \right\}$$

$$= i\omega_m \left\{ -c_m^\dagger c_k c_m - c_k c_m^\dagger c_m \right\}$$

$$= -i\omega_m \delta_{km} c_m \quad \text{(for fermions)}$$

$$= -i\omega_k c_k$$

$$\frac{\partial c_k}{\partial t} = -i\omega_k c_k(t)$$

$$c_k(t) = c_{ks} e^{-i\omega_k t} \quad \text{where } c_k(0) = c_{ks}.$$

- Creation and annihilation operators have the same equal-time commutation relations in any picture.
- $\hat{H}_0 = \sum_k \hbar\omega_k c_k^\dagger c_k$ in either Schroedinger picture or interaction picture.

The time dependence of $|\Psi_I(t)\rangle$

$$|\Psi_I(t)\rangle = e^{iH_0 t/\hbar} |\Psi_s(t)\rangle \quad \text{by definition}$$

$$= e^{iH_0 t/\hbar} e^{-iH(t-t_0)/\hbar} |\Psi_s(t_0)\rangle$$

evolution in Sch. picture

$$= e^{iH_0 t/\hbar} e^{-iH(t-t_0)/\hbar} e^{-iH_0 t_0/\hbar} |\Psi_I(t_0)\rangle$$

$$= \hat{U}(t, t_0) |\Psi_I(t_0)\rangle$$

where

$$\hat{U}(t,t_0) = e^{iH_0 t/\hbar} e^{-iH(t-t_0)/\hbar} e^{-iH_0 t_0/\hbar}$$

/Important! $e^A e^B \neq e^{(A+B)}$ for operators/

$$\frac{\partial \hat{U}}{\partial t} = e^{iH_0 t/\hbar}\left(\frac{i}{\hbar}H_0 - \frac{i}{\hbar}H\right) e^{-iH(t-t_0)/\hbar} e^{-iH_0 t_0/\hbar}$$

$$= -\frac{i}{\hbar} H_1(t)\; e^{+iH_0 t/\hbar} e^{-iH(t-t_0)/\hbar} e^{-iH_0 t_0/\hbar}$$

$$= -\frac{i}{\hbar} H_1(t)\, \hat{U}(t,t_0)$$

Solve by iteration $\Rightarrow$ perturbation theory

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_1(t')\,\hat{U}(t',t_0)\,dt'$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_1(t')\,dt' + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} H_1(t') \int_{t_0}^{t'} H_1(t'')\,\hat{U}(t'',t_0)\,dt'\,dt''$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_1(t')\,dt' + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} H_1(t') \int_{t_0}^{t'} H_1(t'')\,dt'\,dt'' + \left(\frac{-i}{\hbar}\right)^3 \int_{t_0}^{t} H_1(t') \int_{t_0}^{t'} H_1(t'') \int_{t_0}^{t''} H_1(t''')\,\hat{U}(t''',t_0)\,dt'\,dt''\,dt'''$$

= continue the iteration to ∞.

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \cdots \int_{t_0}^{t^{(n-1)}} dt^{(n)}\; H_1(t')\,H_1(t'')\,H_1(t''')\cdots H_1(t^{(n)})$$

Time ordering

(1) The H_1's are time ordered: earlier times stand to the right of later times. (e.g., $t'' < t'$)

Define the time ordered product

$$T\left[H_1(t_1)\,H_1(t_2)\,H_1(t_3)\cdots H_1(t_n)\right] = H_1(t_1')\,H_1(t_2')\,H_1(t_3')\cdots H_1(t_n')$$

where $\{t_1'\,t_2'\,t_3'\cdots t_n'\}$ is the permutation of $\{t_1\,t_2\,t_3\ldots t_n\}$ such that $\{t'\text{'s}\}$ are time ordered.

(!)

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{\cancel{2}\,n} \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\; T\left[H_1(t_1)\,H_1(t_2)\cdots H_1(t_n)\right]$$

= a perturbation expansion.

6d. Adiabatic "switching on"

Assume $H = H_0 + H_1 e^{-\varepsilon|t|}$

and let $\varepsilon \longrightarrow 0$ at the end of the calculations.

Acceptable results must have valid limits as $\varepsilon \longrightarrow 0$.

The initial and final states ,
i.e., as $t \longrightarrow -\infty$ and $+\infty$,
are free particles,
i.e., eigenstates of H_0.

The state experiences the interactions H_1 during the time $-1/\varepsilon \lesssim t \lesssim +1/\varepsilon$.

6e. Gell-Mann & Low theorem

This is a bit of a technicality.

It implies that the limiting process $\varepsilon \longrightarrow 0$ is OK in spite of singularities.

Formally, the state defined by this ratio

$$| \psi_0 (t=0) >_\varepsilon \quad / \quad < \phi_0 | \psi_0 (t=0) >_\varepsilon$$

is well defined as $\varepsilon \longrightarrow 0$;
and it is an eigenstate of the full Hamiltonian, H.

(ϕ_0 means the free particle state at $t = -\infty$.)

The Green's function (or, also called 1 particle matrix element) is $G_{\alpha\beta}(\vec{x}t\,;\,\vec{x}'t')$ where $\alpha\beta$ are spin indices, $\vec{x}\vec{x}'$ are two positions, tt' are two times; is defined by

$$iG_{\alpha\beta}(\vec{x}t;\vec{x}'t') = \frac{\langle\Psi_0|T\left[\psi_{H\alpha}(\vec{x}t)\,\psi^{+}_{H\beta}(\vec{x}'t')\right]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$$

where $|\Psi_0\rangle$ is the ground state in the Heisenberg picture.

(We could require $\langle\Psi_0|\Psi_0\rangle = 1$ but not necessary.)