Chapter 3 : GREEN'S FUNCTIONS AND FIELD THEORY (FERMIONS)

Review

We've developed a nice formal theory for describing many-particle systems. But what can we actually calculate?

What would we *want* to calculate? (Quantities that can be compared to experimental measurements !)

We can't calculate anything exactly --- approximations are a necessary evil.

8. WICK'S THEOREM

Wick's theorem is a formal result that will simplify calculations in perturbation theory.

<u>Statement of the theorem</u>
Any time-ordered product of operators can be expressed as the sum of normal-ordered products multiplied by c-number contractions.

$$
\begin{aligned}
T[\,A\,B\,C\,\ldots\,Z\,] &= N[\,A\,B\,C\,\ldots\,Z\,] \\
&\quad + x_{AB}\; N[\,C\,D\,E\,\ldots\,Z\,] + \text{similar terms} \\
&\quad + x_{AB}\,x_{CD}\; N[\,E\,F\,G\,\ldots\,Z\,] + \text{similar terms} \\
&\quad + \text{all the rest}
\end{aligned}
$$

Why is that useful?
Because the ground-state expectation value of any normal-ordered product is 0.

Preliminaries

(to motivate the importance of Wick's theorem)

The Green's function is the ground state expectation value of a certain operator.
(The operator is the time ordered product of two field operators.)

First consider a general problem

Start with the Heisenberg picture,

$$\langle 0|\, O_H(t)\, |0\rangle.$$

Now write this in the interaction picture
(so that we can apply perturbation theory).

$$\frac{\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle}$$

By the Gell Mann & Low theorem

$$\frac{|\Psi_0\rangle}{\langle \Phi_0 | \Psi_0 \rangle} = \frac{U_\epsilon(t=0;\pm\infty)\,|\Phi_0\rangle}{\langle \Phi_0 | U_\epsilon(t=0;\pm\infty) | \Phi_0 \rangle}$$

(and we take the limit $\epsilon \to 0$).

So

$$\frac{\langle \Psi_0 | \Psi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} = \frac{\langle \Phi_0 | \overbrace{U_\epsilon^\dagger(0,\infty)\, U_\epsilon(0,-\infty)}^{\hat{S}} | \Phi_0 \rangle}{\langle \Phi_0 | U_\epsilon(0,\infty) | \Phi_0 \rangle^* \langle \Phi_0 | U_\epsilon(0,-\infty) | \Phi_0 \rangle}$$

$$\hat{S} = U_\epsilon(\infty,0)\, U_\epsilon(0,-\infty) = U_\epsilon(\infty,-\infty)$$

Similarly

$$\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle = \frac{\langle \Phi_0 | U^\dagger(0,\infty)\, \hat{O}_H(t)\, U_\epsilon(0,-\infty) | \Phi_0 \rangle}{\text{same denominator}}$$

$$\hat{O}_H(t) = U_\epsilon(0,t)\, \hat{O}_I(t)\, U_\epsilon(t,0)$$

$$= \frac{\langle \Phi_0 | U_\epsilon(\infty,0)\, O_I(t)\, U_\epsilon(t,-\infty) | \Phi_0 \rangle}{\text{same denominator}}$$

$$\frac{\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{\langle \Phi_0 | \hat{U}_\epsilon(\infty, t) \hat{O}_I(t) \hat{U}_\epsilon(t, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle}$$

↑ Heisenberg picture

↑ everything here is in the interaction picture

Recall $U_\epsilon(t_2, t_1) = T \exp\left[\frac{-i}{\hbar} \int_{t_1}^{t_2} H_1(t') e^{-\epsilon|t'|} dt' \right]$

∴ we'll have these operators:

$$T(\underbrace{H_1 H_1 \cdots H_1}_{\leftarrow n \text{ terms} \rightarrow}) \hat{O}_I(t) \, T(\underbrace{H_1 H_1 \cdots H_1}_{\leftarrow m \text{ terms} \rightarrow})$$

which we can replace by

$$T(\underbrace{H_1 H_1 \cdots H_1 H_1}_{\leftarrow n+m \text{ terms} \rightarrow} \hat{O}_I(t))$$

Hence $\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle / \langle \Psi_0 | \Psi_0 \rangle$

$$= \frac{1}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle} \sum_{\nu=0}^{\infty} \left(\frac{-i}{\hbar} \right)^\nu \frac{1}{\nu!}$$

$$\int dt_1 \, e^{-\epsilon|t_1|} \int dt_2 \, e^{-\epsilon|t_2|} \cdots \int dt_\nu \, e^{-\epsilon|t_\nu|}$$

$$\langle \Phi_0 | T\left[H_1(t_1) H_1(t_2) \cdots H_1(t_\nu) \hat{O}_H(t) \right] | \Phi_0 \rangle$$

where $\int dt_i$ means $\int_{-\infty}^{\infty} dt_i$

Similarly for the 1-particle Green's function

$$iG_{\alpha\beta}(x,t) = \frac{\langle \Phi_0 | T[\psi_\alpha(x)\, \psi^+_\beta(y)] | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle}$$

$$= \sum_{\nu=0}^{\infty} \left(\frac{-i}{\hbar}\right)^\nu \frac{1}{\nu!} \int dt_1\, dt_2 \ldots dt_\nu$$

$$\frac{\langle \Phi_0 | T[H_1(t_1)\, H_1(t_2) \ldots H_1(t_\nu)\, \psi_\alpha(x)\, \psi^+_\beta(y)] | \Phi_0 \rangle}{\langle \Phi_0 | \hat{S} | \Phi_0 \rangle}$$

x means $(\vec{x}, t_x)$
y means $(\vec{y}, t_y)$

$\Longrightarrow$ perturbation expansion

$$iG_{\alpha\beta}(x,y) = \Big\{ i\, G^0_{\alpha\beta}(x,y)$$

$$+ \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{\infty} dt_1\, \langle \Phi_0 | T[H_1(t_1)\, \psi_\alpha(x)\, \psi^+_\beta(y)] \Phi_0 \rangle$$

$$+ \ldots \Big\} \Big/ \langle \Phi_0 | \hat{S} | \Phi_0 \rangle.$$

all operators on right of "="
are interaction picture operators.

So, the problem reduces to calculations of

$$\langle \Phi_0 | T[\psi\psi\psi \ldots \psi\; \psi^+ \psi^+ \psi^+ \ldots \psi^+] | \Phi_0 \rangle$$

in the interaction picture.

Time ordering and normal ordering

We already know time ordering.

★ $T(A' B' C' D' \ldots) = (-1)^P \times (A\ B\ C\ D \ldots)$
where $\{A\ B\ C\ D \ldots\}$ are in time order.

What is normal ordering?

Assume (as is usually the case) that the field operator has both annihilation terms and creation terms.

Example: In relativistic QED, $\psi(x)$ annihilates electrons and creates positrons.

Example: In the quantum theory of metals, $\psi(x)$ annihilates electrons above the Fermi energy ("particles") and creates "holes" below the Fermi energy.

Example: In the the nuclear shell model, $\psi(x)$ annihilates nucleons above the filled shells ("particles") and creates holes in the filled shells.

So, we can write

$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$
= annihilation part plus creation part;
note $\psi^{(+)}(x)|\Phi_0\rangle = 0$.

Also,

$\psi^\dagger(x) = \psi^{(+)\dagger}(x) + \psi^{(-)\dagger}(x)$
= creation part plus annihilation part;
note $\psi^{(-)\dagger}(x)|\Phi_0\rangle = 0$.

★ A product of field operators is in normal order if all the annihilation operators stand to the right of all creation operators.

★ $N(A' B' C' \ldots) = (-1)^P \times (A\ B\ C \ldots)$ where $\{A\ B\ C \ldots\}$ are in normal order.

Theorem. The expectation value in Φ_0, of a normal ordered product , is 0.

Wick's theorem

T(U V W … X Y Z)

= N(U V W … X Y Z) + all possible pairs of contractions.

See FW for the general proof.

Proof by examples (assuming fermions)

Suppose U V W are annihilation parts at later times than X Y Z which are all creation parts.

Then

Ξ = T(U V W X Y Z) = U V W X Y Z.

But this is not in normal order.

Move X to the left using the commutation relations.

Ξ = U V W X Y Z = UV ({W,X} – XW) YZ

= – UVXWYZ + c(W,X) UVYZ

(the contraction is a c number)

In the first term move X to the left; in the second term move Y to the left.

Ξ = – (– UXVWYZ + c(V,X) UWYZ)

+ c(W,X) (– UYVZ + c(V,Y) UZ)

keep going, always moving creation parts to the left

Ξ = – XUVWYZ + c(U,X) VWYZ

–c(V,X) (–UYWZ + c(W,Y) UZ)

–c(W,X) (–YUVZ + c(Y,V) UZ)

+c(W,X)c(V,Y) (– ZU + c(U,Z))

until all the terms are in normal order.

Ξ = – XYZUVW + c(U,X) YZVW + many similar

–c(U,X) c(W,Y) ZV + many similar

+c(W,X) c(V,Y) c(U,Z) + many similar

= N(UVWXYZ) + all possible pairs of contractions.

What are the contractions?

In the interaction picture,

$$\hat{\psi}(\vec{x}t) = \sum_{\vec{k}\lambda} \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{V}} u_\lambda e^{-i\omega_k t} [\underbrace{\theta(k-k_F)\, a_{\vec{k}\lambda}}_{\psi^{(+)}} + \underbrace{\theta(k_F-k)\, b^\dagger_{-\vec{k}\lambda}}_{\psi^{(-)}}] \qquad \psi^{(\pm)} \begin{cases} \text{annihilation} \\ \text{creation} \end{cases}$$

$$\hat{\psi}^\dagger(\vec{x},t) = \sum_{\vec{k}\lambda} \frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{V}} u^\dagger_\lambda e^{i\omega_k t} [\underbrace{\theta(k-k_F)\, a^\dagger_{\vec{k}\lambda}}_{\psi^{(+)\dagger}} + \underbrace{\theta(k_F-k)\, b_{-\vec{k}\lambda}}_{\psi^{(-)\dagger}}] \qquad \psi^{(\pm)\dagger} \begin{cases} \text{creation} \\ \text{annihilat.} \end{cases}$$

- $\{\psi^{(\pm)}(x), \psi^{(\pm)}(y)\} = 0$ because $\{a, a\} = \{a, b^\dagger\} = \{b^\dagger, a\} = \{b^\dagger, b^\dagger\} = 0$
- $\{\psi^{(\pm)\dagger}(x), \psi^{(\pm)\dagger}(y)\} = 0$ similarly
- $\{\psi^{(+)}(x), \psi^{(-)\dagger}(y)\} = 0$ because $\{a, b\} = 0$
- $\{\psi^{(-)}(x), \psi^{(+)\dagger}(y)\} = 0$ because $\{b^\dagger, a^\dagger\} = 0$

so there are only two nonzero contractions (Eq. (8.27))

What are the contractions?

$$c\left(\psi_\alpha^{(+)}(x),\ \psi_\beta^{(+)\dagger}(y)\right) = T\left[\psi_\alpha^{(+)}(x)\,\psi_\beta^{(+)\dagger}(y)\right] - N\left[\psi_\alpha^{(+)}(x)\,\psi_\beta^{(+)\dagger}(y)\right]$$

$\boxed{\text{If } t_x > t_y}$ then $c = \psi_\alpha^{(+)}(x)\,\psi_\beta^{(+)\dagger}(y) + \psi_\beta^{(+)\dagger}(y)\,\psi_\alpha^{(+)}(x)$

$$= \sum_{\vec{k}\lambda} \sum_{\vec{k}'\lambda'} \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{V}} u_{\lambda\alpha} e^{-i\omega_k t_x} \frac{e^{-i\vec{k}'\cdot\vec{y}}}{\sqrt{V}} u^\dagger_{\lambda'\beta} e^{i\omega_{k'} t_y}\, \theta(k-k_F)\,\theta(k'-k_F)\, \underbrace{\left\{ a_{\vec{k}\lambda} a^\dagger_{\vec{k}'\lambda'} + a^\dagger_{\vec{k}'\lambda'} a_{\vec{k}\lambda} \right\}}_{\delta_{\lambda\lambda'}\,\delta(\vec{k},\vec{k}')}$$

$$= \delta_{\alpha\beta} \sum_{\vec{k}} \frac{1}{V} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} e^{-i\omega_k (t_x - t_y)}\, \theta(k-k_F)$$

$$= i\,G^0_{\alpha\beta}(x,y) \quad \text{for } t_x > t_y .$$

$\boxed{\text{If } t_x < t_y}$ then $c = -\psi_\beta^{(+)\dagger}(y)\,\psi_\alpha^{(+)}(x) + \psi_\beta^{(+)\dagger}(y)\,\psi_\alpha^{(+)}(x) = 0$

Result $\quad c\left(\psi_\alpha^{(+)}(x),\ \psi_\beta^{(+)\dagger}(y)\right) = \begin{cases} i\,G^0_{\alpha\beta}(xy) & \text{for } t_x > t_y \\ 0 & \text{for } t_x < t_y \end{cases}$

Similarly $\quad c\left(\psi_\alpha^{(-)}(x),\ \psi_\beta^{(-)\dagger}(y)\right) = \begin{cases} 0 & \text{for } t_x > t_y \\ i\,G^0_{\alpha\beta}(xy) & \text{for } t_x < t_y \end{cases}$

$\boxed{\Rightarrow}\quad c\left(\psi_\alpha(x),\ \psi_\beta^\dagger(y)\right) = i\,G^0_{\alpha\beta}(xy) \quad$ (Equation 8.29)