Chapter 4 : FERMI SYSTEMS

Review

11. THE IMPERFECT FERMI GAS

Helium

Helium has two stable isotopes

A He-4 atom is a boson (2 proton, 2 neutrons, 2 electrons; S = 0). At low temperatures, there is not enough thermal energy to excite the atom, so it is just an inert particle. At pressure = 1 atm:
For T < 4.23 K, He-4 is a liquid.;
For T < 2.17 K, He-4 is a superfluid.

A He-3 is a fermion (2 protons, 1 neutron, 2 electrons; S = ½). Natural abundance: 1 in 10^4.
At pressure = 1 atm:
For T < 3.19 K, He-3 is a liquid.;
For T < 0.002 K, He-3 is a superfluid.
(Nobel prize 1996 Lee Osheroff Richardson)

Landau's theory of Fermi liquids

If the interparticle potential is 0, then N particles form an ideal Fermi gas.
The energies of the single-particle states are $\hbar^2k^2/2m$.

Now turn on the interparticle potential slowly (adiabatically). Similar to the GellMann & Low theorem, the particles will remain, to a good approximation, in one-particle states, occupying the states up to some maximum energy (the Fermi energy).

The energies of the single particle states will not be $\hbar^2k^2/2m$.
The excitations will be unstable states with energies above the Fermi energy.

We use this language taken from quantum electrodynamics.

If the interparticle potential is 0 then an additional particle propagates freely. The propagator is $G^0(x,y)$,

$$\bar{G}^0 = \frac{\Theta(k-k_F)}{\omega-\omega_k^0+i\eta} + \frac{\Theta(k_F-k)}{\omega-\omega_k^0-i\eta}$$

and we call this the "bare propagator".

When the interparticle potential is acting then the particle collides with particles below the Fermi energy. Then the propagator is G(x,y),

$$\bar{G} = \frac{1}{\omega-\omega_k^0+\Sigma^*(k)} \cong \frac{1}{\omega-\omega_k+i\gamma_k}$$

and we call this the "dressed propagator".

What is the interparticle potential?

There is a weak attraction (like a van der Waals force) at large distances, and a strong repulsion (like a hard sphere potential)* at short distances.

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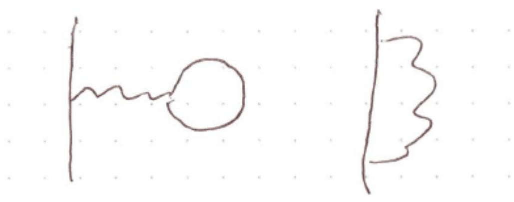
* By the way, the nucleon-nucleon potential is similar.

For normal (i.e., not superfluid) He-3 we are mainly interested in the effects of the short-range repulsion.

But for that we cannot use perturbation theory, because the interaction is not small.

Galitskii's method

We had, in perturbation theory,



Now, replace the potential by the scattering amplitude.

The result is Galitskii's integral equation.

Results

$$G(\vec{p}, p_0) \cong \left[p_0 - \frac{\hbar p^2}{2m} - \Sigma^*_{(1)}(\vec{p}, p_0) - \Sigma^*_{(2)}(\vec{p}, p_0) \right]^{-1}$$

where

$$\hbar \Sigma^*_{(1)} = -i \int \frac{d^4k}{(2\pi)^4} G^0(k) \frac{4\pi a \hbar^2}{m} e^{ik_0 \eta}$$

$$= \frac{\hbar^2 k_F^2}{m} \frac{2 k_F a}{3\pi}$$

Here a = the s-wave scattering length;

The scattering amplitude $f(\vec{k}, \vec{k}') \approx -a$ in limit $|\vec{k}| \to 0$.

For hard sphere scattering, a = radius of the sphere.

We are expanding G in powers of $k_F a$.

A dilute liquid has $k_F a \ll 1$.

Now we need $\Sigma^*_{(2)}(\vec{p}, p_0)$.

$$\hbar \Sigma^*_{(2)}(\vec{p}, p_0) = \frac{\hbar^2}{m} 16\pi^2 a^2 \int \frac{d^3k \, d^3k'}{(2\pi)^6}$$

$$\left\{ \frac{A}{p_0 - \frac{\hbar p^2}{2m} + \frac{\hbar}{m}\left[\frac{(\vec{p}-\vec{k})^2}{4} - k'^2\right] + i\epsilon} + \frac{B}{p_0 - \frac{\hbar p^2}{2m} + \frac{\hbar}{m}\left[\frac{(\vec{p}-\vec{k})^2}{4} - k'^2\right] - i\epsilon} + \mathrm{P.V.} \frac{\theta(k_F - k)}{\frac{\hbar}{m}\left[\frac{(\vec{p}-\vec{k})^2}{4} - k'^2\right]} \right\} \quad (11.58)$$

$$A = \theta(k_F - k)\, \theta\left(\left|\tfrac{1}{2}\vec{P} + \vec{k}'\right| - k_F\right) \theta\left(\left|\tfrac{1}{2}\vec{P} - \vec{k}'\right| - k_F\right)$$

$$B = \theta(k - k_F)\, \theta\left(k_F - \left|\tfrac{1}{2}\vec{P} + \vec{k}'\right|\right) \theta\left(k_F - \left|\tfrac{1}{2}\vec{P} - \vec{k}'\right|\right)$$

and $\vec{P} = \vec{p} + \vec{k}$

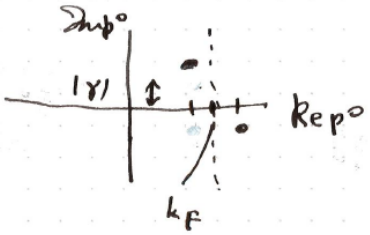
I wasn't able to find an exact evaluation of these integrals!

PHYSICAL QUANTITIES

① Lifetime of the quasiparticle

$$\hbar\gamma_p = \frac{\hbar^2 k_F^2}{2m}\,\frac{2}{\pi}(k_F a)^2\left(\frac{k_F-p}{k_F}\right)^2$$

$$\times\ \mathrm{sign}(k_F - p)$$



As $p \to k_F$, $\gamma_p \to 0$ so the excitation is long lived.

These are the "quasi particles", relevant for low temperature.

② Excitation energy spectrum

First, the Fermi energy

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m}\left[1 + \frac{4}{3\pi}(k_F a) + \frac{4}{15\pi^2}(11 - 2\ln 2)(k_F a)^2\right]$$

from (11.58) by Galitskii.

i.e., $\hbar\Sigma^*_{(2)}\left(\vec{p}, \frac{\hbar p^2}{2m}\right)$ with $\frac{\hbar^2 p^2}{2m} = \epsilon_F$.

Now $\epsilon_{\vec{p}} = \epsilon_F + \frac{\hbar^2 p^2}{2m} - \epsilon_F + \hbar\Sigma^*_{(1)} + \hbar\Sigma^*_{(2)}$

For states close to the Fermi surface

$$\epsilon_p = \epsilon_F + (p - k_F)\left(\frac{\partial \epsilon_p}{\partial p}\right)_{k_F} + \cdots$$

Define effective mass m_* by

$$\epsilon_p = \frac{\hbar^2 p^2}{2m_*} \Rightarrow \left(\frac{\partial \epsilon_p}{\partial p}\right)_{k_F} = \frac{\hbar^2 k_F}{m_*}$$

Thus

$$\frac{1}{m_*} = \frac{1}{\hbar^2 k_F} \underbrace{\left(\frac{\partial \epsilon_p}{\partial p}\right)_{k_F}}_{\text{slope of } \epsilon_p \text{ at } k_F}$$

Result

$$\frac{m_*}{m} = 1 + \frac{8}{15\pi^2}(7\ln 2 - 1)(k_F a)^2$$

neglecting $O(k_F a)^3$.

③ Ground state energy

$$\frac{E}{N} = \frac{\hbar^2 k_F^2}{2m}\Big[\frac{3}{5} + \frac{2}{3\pi} k_F a$$

$$+ \frac{4}{35\pi^2}(11 - 2\ln 2)(k_F a)^2$$

$$+ 0.23 (k_F a)^3 + \cdots \Big]$$

(an extra calculation)

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Fermi systems with long scattering lengths

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A. Low densities (dilute): $k_F|a|\ll 1$

At low densities, $k_F|a|\ll 1$, gaps are small and have little effect on the total energy of the system. Expanding the effective scattering amplitude of Eq. (1) in the small quantity $k_F a$, the energy per particle is obtained from Eq. (6) by summing over momenta of the two interacting particles

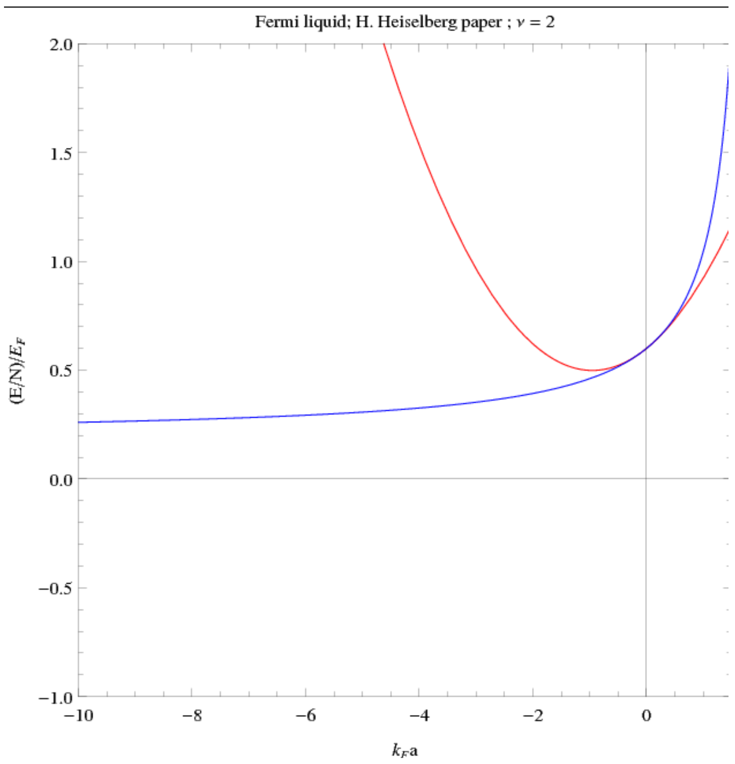
$$\frac{E}{N}=E_F\left\{\frac{3}{5}+(\nu-1)\frac{2}{3\pi}k_F a+(\nu-1)\frac{4(11-2\ln 2)}{35\pi^2}\times(k_F a)^2+\mathcal{O}((k_F a)^3)\right\}. \tag{7}$$

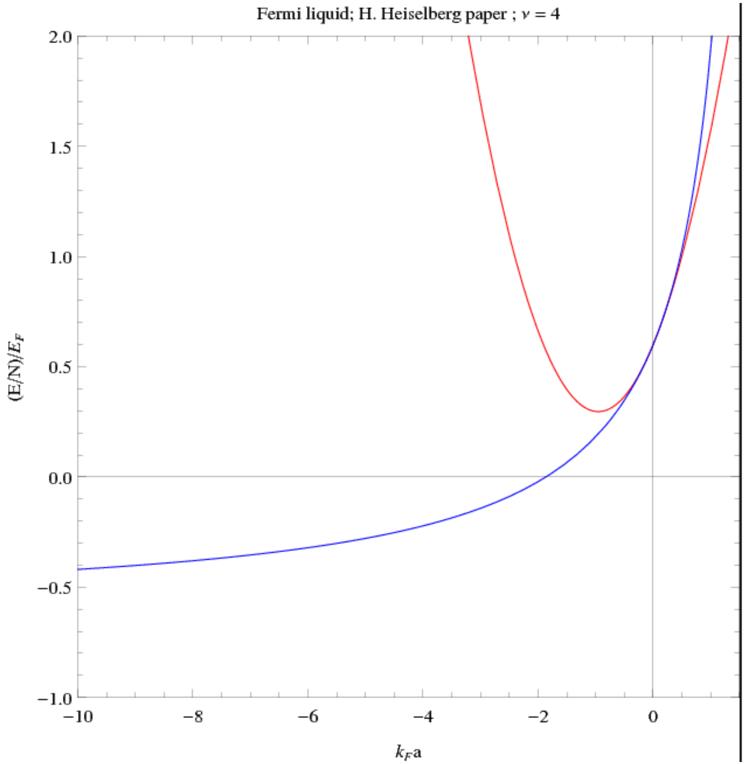
$$\frac{E}{N}\simeq E_F\left[\frac{3}{5}+\frac{(\nu-1)\frac{2}{3\pi}k_F a}{1-\frac{6}{35}\pi(11-2\ln 2)k_F a}\right]. \tag{11}$$

This expression is valid for dilute systems, where it reproduces Eq. (7), and approximately valid within the Galitskii ladder resummation at intermediate densities, $R\ll k_F^{-1}\ll|a|$, where it reduces to

$$\frac{E}{N}=E_F\left[\frac{3}{5}-(\nu-1)c_1\right]=E_F c_1(\nu_c-\nu), \tag{12}$$

with $c_1=35/9(11-2\ln 2)\simeq 0.40$ and $\nu_c=1+3/5c_1\simeq 2.5$. Both the attractive and the kinetic part of the energy per particle are proportional to the Fermi energy at these intermediate energies as found for the gaps above.


Fermi liquid; H. Heiselberg paper ; $\nu = 2$
(E/N)/E_F
k_Fa
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−10
−8
−6
−4
−2
0


Fermi liquid; H. Heiselberg paper ; $\nu = 4$
(E/N)/E_F
k_Fa
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−10
−8
−6
−4
−2
0