Chapter 4 : FERMI SYSTEMS

Review

We studied the degenerate electron gas (or, *jellium*) in Section 3, and homework problem 6. There the discussion was limited to first order perturbation theory.

Now using QFT we'll consider a more accurate result.

12. The DEGENERATE ELECTRON GAS

Consider particles that interact by a spin-independent potential,

$$V(\vec{x},\vec{x}')_{\lambda\lambda',\mu\mu'} = V(\vec{x}-\vec{x}')\,\delta_{\lambda\lambda'}\,\delta_{\mu\mu'}$$

$$U_0(xx')_{\lambda\lambda'\mu\mu'} = V(\vec{x}-\vec{x}')\,\delta_{\lambda\lambda'}\,\delta_{\mu\mu'}\,\delta(t-t')$$

This is

.

Then

$$\langle\hat{V}\rangle = \frac{1}{2}\int d^3x\, d^3x'\; V(\vec{x}-\vec{x}')\,\langle\Phi_0|\psi^\dagger_\alpha(\vec{x})\,\psi^\dagger_\beta(\vec{x}')\,\psi_\beta(\vec{x}')\,\psi_\alpha(\vec{x})|\Phi_0\rangle$$

Now, let $n(\vec{x}) = \psi^\dagger_\alpha(\vec{x})\,\psi_\alpha(\vec{x})$

$$\langle\hat{V}\rangle = \frac{1}{2}\int d^3x\, d^3x'\; V(\vec{x}-\vec{x}')\left\{\langle n(\vec{x})\,n(\vec{x}')\rangle - \delta^3(\vec{x}-\vec{x}')\,\langle n(\vec{x})\rangle\right\}$$

Or, let $\tilde{n}(\vec{x}) = n(\vec{x}) - \langle n(\vec{x}) \rangle$.

For a uniform system,

$$\langle n(\vec{x}) \rangle = n \quad (= N/V)$$

Then

$$\langle \hat{V} \rangle = \frac{1}{2}\int d^3x\, d^3x' \; V(\vec{x}-\vec{x}') \left[\underbrace{\langle \tilde{n}(\vec{x})\, \tilde{n}(\vec{x}') \rangle}_{\text{"density correlation function"}} - n^2 - \delta^3(x-x')\, n \right]$$

Define a time ordered density correlation function

$$iD(x,x') = \langle \Psi_0 | \, T [\underbrace{\tilde{n}(x)}_{n_H\text{: Heisenberg picture}} \tilde{n}(x')] \, | \Psi_0 \rangle$$

also

$$iD_0(x,x') = \langle \Phi_0 | \, T [\underbrace{\tilde{n}_I(x)}_{\text{interaction picture}} \tilde{n}_I(x')] \, | \Phi_0 \rangle$$

Then

$$\langle \hat{V} \rangle = \underbrace{\langle \Phi_0 | \hat{V} | \Phi_0 \rangle}_{\text{This part was calculated in Section 3.}} + \frac{1}{2}\int d^3x\, d^3x' \; V(\vec{x}-\vec{x}') \left[iD(\vec{x}'t, \vec{x}t) - iD^0(\vec{x}'t, \vec{x}t) \right]$$

Thus the ground state energy is

$$E = \underbrace{\langle \Phi_0 | \hat{H} | \Phi_0 \rangle}_{\text{known from Section 3}} + \frac{1}{2}\int_0^1 \frac{d\lambda}{\lambda}\, d^3x\, d^3x' \; \lambda V(\vec{x}-\vec{x}') \left[iD^\lambda(\vec{x}'t; \vec{x}t) - iD^0(\vec{x}'t, \vec{x}t) \right]$$

using a trick for calculating the potential energy (Eq. 7.30)

The second term is called the "correlation energy." In momentum space,

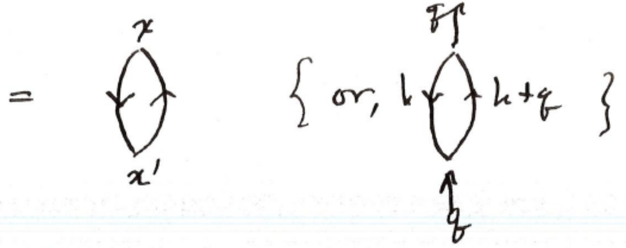
$$\underset{\text{(Volume)}}{E_{corr}} = \frac{V}{2}\int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4q}{(2\pi)^4}\, \lambda V(\vec{q}) \left[iD^\lambda(\vec{q},\omega) - iD^0(\vec{q},\omega) \right]$$

is for

$iD^0(x,x')$ is for ~~non~~ noninteracting particles;

By Wick's theorem

$$iD^0(xx') = \underset{\text{spins}}{(2S+1)}\, G^0(xx')\, G^0(x'x)$$



which is the lowest order polarization insertion

So

$$D^0(x,x') = \hbar\, \Pi^0(x,x')$$

Same is true for the interacting theory,

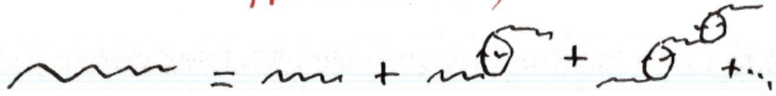
$$D(x,x') = \hbar\, \Pi(x,x')$$

Dyson's equation: $\int d^4x_1\, U_0(xx_1)\, \Pi(x_1x')$

$$= \int d^4x_1\, U(xx_1)\, \underbrace{\Pi^*(x,x')}_{\text{sum of proper polarization insertions.}}$$

The Ring Approximation

(FW say this is sometimes called the "random phase approximation")



$$E_{ring} = \sum_{n=2}^{\infty} E_{ring,n}$$

$$= \frac{i\hbar V}{2}\int_0^1 \frac{d\lambda}{\lambda}\int \frac{d^4q}{(2\pi)^4} \sum_{n=2}^{\infty}\left[\lambda U_0(q)\Pi^0(q)\right]^n$$

$$= \text{------} \int \frac{d^4q}{(2\pi)^4} \frac{(\lambda U_0 \Pi^0)^2}{1-\lambda U_0 \Pi^0}$$

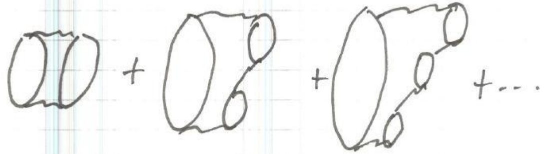
$$= \text{------} \int \frac{d^4q}{(2\pi)^4} \lambda U_0 \Pi^0 \; \underline{U^{\lambda}_{RING}(q)} \; \Pi^0$$

defines the effective interacti

λV_0 = bare interaction

U^{λ}_{RING} = effective interaction

So, we have then leading contribution to the correlation energy

$E_{RING} =$ 

Results for $\Pi^0(\vec{q}, q_0)$ where $\nu = \frac{\hbar q_0}{2(\hbar^2 k_F^2/2m)}$

$$\mathrm{Re}\,\Pi^0(\vec{q},q_0) = \frac{2mk_F}{\hbar^2}\frac{1}{4\pi^2}\left\{-1 + \frac{1}{2q}\left[1-\left(\frac{\nu}{q}-\frac{q}{2}\right)^2\right]\ln\left|\frac{1+(\nu/q-q/2)}{1-(\nu/q-q/2)}\right| - \frac{1}{2q}\left[1-\left(\frac{\nu}{q}+\frac{q}{2}\right)^2\right]\ln\left|\frac{1+(\nu/q+q/2)}{1-(\nu/q+q/2)}\right|\right\}$$

and $\mathrm{Im}\,\Pi^0(\vec{q},q_0)$ is more complicated.

The ring approximation for the ~~tot~~ correlated energy is

$$E_{Ring} = \frac{i\hbar V}{2}\int \frac{d^4q}{(2\pi)^4}\int_0^1 \frac{d\lambda}{\lambda} \frac{\left[\lambda U_0(q)\,\Pi^0(q)\right]^2}{\left[1-\lambda U_0(q)\,\Pi^0(q)\right]}$$

$$= \frac{-i\hbar V}{2}\int \frac{d^4q}{(2\pi)^4}\left\{\ln\left[1-U_0(q)\,\Pi^0(q)\right] + U_0(q)\,\Pi^0(q)\right\}$$

$$= \frac{Ne^2}{2a_0}\,\epsilon_{RING}$$ for V_{ee} Coulomb interaction.

$\frac{e^2}{2a_0}$ = the Rydberg energy.

$$\epsilon_{RING} = \frac{2}{\pi^2}(1-\ln 2)\ln r_s + \text{constant}$$

in the limit $r_s \to 0$.

From Feynman, Statistical Mechanics, page 264

$$\frac{E}{N} = \frac{2.22}{r_s^2} - \frac{0.916}{r_s} + 0.0622 \ln r_s - 0.096 + O(r_s) \quad \text{Rydbergs}$$

r_s = average spacing in units of Bohr radius

$r_s = r_0/a_B$

Homework Problem #16.

12e. The effective interaction

$$U_r(\mathbf{q},0) = \frac{4\pi e^2}{q^2 + (4\alpha r_s/\pi)\, k_F^2\, g(q/k_F)}$$

$$g(x) = \frac{1}{2} - \frac{1}{2x}\left(1 - \tfrac{1}{4}x^2\right) \ln\left|\frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}x}\right|$$

$$U_r(\mathbf{q},0) \underset{r_s \to 0}{\approx} \frac{4\pi e^2}{q^2 + (4\alpha r_s/\pi)\, k_F^2}$$

$$V_r(\mathbf{x}) \approx e^2\, e^{-q_{TF} x}\, x^{-1}$$

The e^2/r Coulomb potential is screened, with screening length $1/q_{TH}$, by the other charges (positive and negative) in the neighborhood of the electrons.