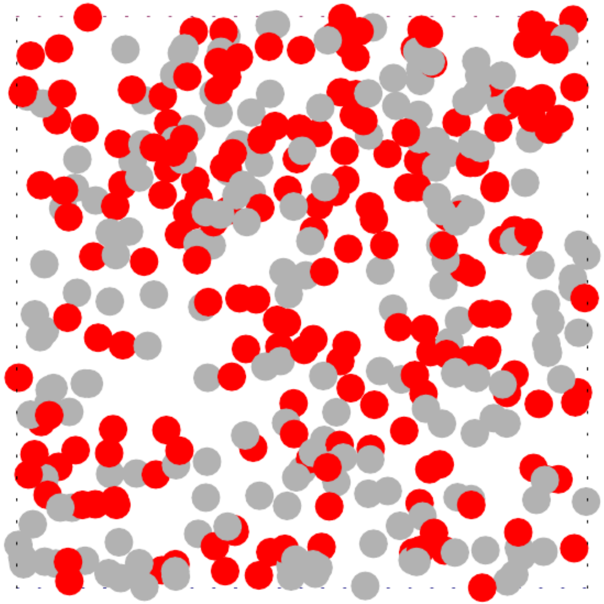
Chapter 11 : Nuclear Matter

nuclear matter



Section 40: INDEPENDENT PARTICLE (FERMI GAS) MODEL

We've done some of this before. This is the model:

/1/ Single particle wave functions are

$$\varphi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}}$$

In fact these are solutions of the Hartree Fock equation because of the translation invariance.

/2/ Use isospin states to distinguish protons and neutrons .

$$\phi(\vec{x}) = \varphi_{\vec{k}}(\vec{x})\, \eta_\lambda\, \zeta_\rho$$

(η_λ: spin state; ζ_ρ: isospin state)

Spin : Basis states are $\eta_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$ and $\eta_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$

Spin operator $= \frac{\hbar}{2}\vec{\sigma}$

$$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Isospin: Basis states are

$$\zeta_p = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \zeta_n = \begin{pmatrix}0\\1\end{pmatrix}$$

Isospin operator $\vec{\tau} = \vec{\sigma}$.

/3/ Second quantization

The nucleon field

$$\psi(\vec{x}) = \sum_{\vec{k}} \sum_{\lambda} \sum_{\rho} \varphi_{\vec{k}}(\vec{x})\, \eta_\lambda \zeta_\rho\, a_{\vec{k}\lambda\rho}$$

where

$$\{a_{\vec{k}\lambda\rho}, a^\dagger_{\vec{k}'\lambda'\rho'}\} = \delta_{Kr}(\vec{k},\vec{k}')\, \delta_{\lambda\lambda'}\, \delta_{\rho\rho'}$$

$a_{\vec{k}\lambda 1}$ annihilates a proton w/ $(\vec{k},\lambda)$

$a_{\vec{k}\lambda 2}$ annihilates a neutron w/ $(\vec{k},\lambda)$

{Proton and neutrons are not identical. FW point out that is there is no physical that converts p < -- > n, then we can use either $\{a_p, a_n^+\} = 0$ or $[a_p, a_n^+]=0$; they'll give the same answers.}

/4/ Estimate the ground state energy using first order perturbation theory.

$$E \le \langle F|\hat{H}|F\rangle$$

which is a bound by the Rayleigh Ritz variational principle.

$|F\rangle$ = filled states below ϵ_F.

$$\langle F|\hat{H}|F\rangle = \langle F|\hat{H}_0 + \hat{H}_1|F\rangle = E_0 + E_1$$

$$E_0 = 2\times 2\times \sum_{\vec{k}}^{k_F} \frac{\hbar^2 k^2}{2m}$$

$$E_1 = \frac{1}{2}\sum_{k_1 k_2 k_3 k_4} \langle k_1 k_2|V|k_3 k_4\rangle \langle F|a^+_{k_1} a^+_{k_2} a_{k_4} a_{k_3}|F\rangle$$

← 2-body potential; function of $k_1 \cdots k_4$

← matrix element in Fock space

Note: $k_i = \{\vec{k}_i, \lambda_i, \rho_i\}$

The matrix element

$$\langle F|a^+_{k_1} a^+_{k_2} a_{k_4} a_{k_3}|F\rangle = \delta_{k_1 k_3}\delta_{k_2 k_4} - \delta_{k_1 k_4}\delta_{k_2 k_3}$$

↳ i.e., $\delta_{Kr}(\vec{k}_1,\vec{k}_3)\,\delta_{\lambda_1\lambda_3}\,\delta_{\rho_1\rho_3}$ etc

Thus

$$E_1 = \frac{1}{2}\sum_{k_1 k_2}\{\langle k_1 k_2|V|k_1 k_2\rangle - \langle k_1 k_2|V|k_2 k_1\rangle\}$$

(direct term + exchange term)

/5/ Choice of a 2-body potential; based on the review of the nucleon nucleon force.

Today we'll consider this choice:

V = Wigner term + Majorana (exchange) term

$$V = V_0(|\mathbf{x}_1 - \mathbf{x}_2|) \{ a_W + a_M P_M \}$$

where P_M is the exchange operator,

$$P_M f(\mathbf{x}_1,\mathbf{x}_2) = f(\mathbf{x}_2,\mathbf{x}_1);$$

also, $V_0(r)$ is attractive, nonsingular, and spin independent.

Calculation of E_0

E_0 =

$$E_0 = 4 \sum_{\vec{k}}^{k_F} \frac{\hbar^2 k^2}{2m}$$

$$= 4 \frac{V}{(2\pi)^3} \int^{k_F} d^3k \frac{\hbar^2 k^2}{2m}$$

$$= \frac{4V}{(2\pi)^3} \frac{\hbar^2}{2m} \frac{4\pi k_F^5}{5}$$

Recall the Fermi momentum:

$$A = 4 \sum_{k}^{k_F}$$

$$= \frac{4V}{(2\pi)^3} \frac{k_F^3}{3}$$

Therefore

$$\frac{E_0}{A} = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}$$

Calculation of the "direct term" of E_1

V_{direct} =

$$V_{direct} = \frac{1}{2} \sum_{k_1 k_2} \langle k_1 k_2 | V | k_1 k_2 \rangle$$

$r = |\vec{x}_1 - \vec{x}_2|$

$$\langle \ldots \rangle = \int \frac{d^3x_1}{V} \frac{d^3x_2}{V} e^{-ik_1 \cdot x_1} e^{-ik_2 \cdot x_2} V_0(r) \cdot \left\{ a_W e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} + a_M e^{ik_1 \cdot x_2} e^{ik_2 \cdot x_1} \right\}$$

$$\sum_{\substack{\lambda_1 \lambda_2 \\ \rho_1 \rho_2}} \eta^+_{\lambda_1} \eta^+_{\lambda_2} \zeta^+_{\rho_1} \zeta^+_{\rho_2} \eta_{\lambda_1} \eta_{\lambda_2} \zeta_{\rho_1} \zeta_{\rho_2}$$

Spin isospin line $= \sum_{\substack{\lambda_1 \lambda_2 \\ \rho_1 \rho_2}} \cdot 1 = 2^4 = 16$

$$\langle \ldots \rangle = 16 \int \frac{d^3x_1 \, d^3x_2}{V \cdot V} V_0(r) \left\{ a_W + a_M e^{-i(\vec{k}_1 - \vec{k}_2) \cdot (\vec{x}_1 - x_2)} \right\}$$

$$\langle \ldots \rangle = 16 \int \frac{d^3x_1 \, d^3x_2}{V \cdot V} V_0(r) \left\{ a_W + a_M e^{-i(\vec{k}_1 - \vec{k}_2) \cdot (\vec{x}_1 - \ldots} \right.$$

Let $\vec{z} = \vec{x}_1 - \ldots$

Note $|\vec{z}| = r$

$$= \frac{16}{V} \left\{ a_W \int d^3z \, V_0(r) + a_M \int d^3z \, e^{-i(\vec{k}_1 - \vec{k}_2) \cdot \vec{z}} V_0(\ldots \right.$$

$$V_{direct} = \frac{1}{2} \frac{V^2}{(2\pi)^3} \int d^3k_1 \, d^3k_2 \, \frac{16}{V} \left\{ a_W \int V_0 + a_M \int e^{-i\vec{q} \cdot \vec{z}} \ldots \right.$$

$$\int^{k_F} d^3k_1 = \frac{4}{3}\pi k_F^3$$

$$\int^{k_F} d^3k_1 \, e^{-i\vec{k}_1 \cdot \vec{z}} V_0(r) = 2\pi \int_0^{k_F} k_1^2 dk_1 \underbrace{\int_{-1}^{1} e^{-ik_1 r u} du}_{\frac{1}{ik_1 r} 2i \sin(k_1 r)} V_0(r)$$

$$= 2\pi \int_0^{k_F} 2 j_0(k_1 r) k_1^2 dk_1 \, V_0(r)$$

$$= \frac{4}{3}\pi k_F^3 \, \frac{3 j_1(k_F r)}{k_F r} V_0(r)$$

Spherical Bessel functions:

$$j_0(x) = \frac{\sin x}{x} \; ; \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$V_{direct} = \frac{k_F^3 A}{48\pi^2}\left\{16\, a_W \int V_0\, d^3z + 16\, a_M \int V_0 \left[\frac{3 j_1(k_F r)}{k_F r}\right]^2 d^3z\right\}$$

Calculation of the "exchange term" of E_1

$V_{exchange}$ =

$$V_{exchange} = \frac{-1}{2}\sum_{k_1 k_2} \langle k_1 k_2 | V | k_2 k_1 \rangle$$

$$= \int \frac{d^3x_1}{V}\frac{d^3x_2}{V}\, e^{-ik_1 x_1} e^{-ik_2 x_2}\, V_0(r) \left\{ a_W\, e^{ik_2 x_1} e^{ik_1 x_2} + a_M\, e^{ik_2 x_2} e^{ik_1 x_1} \right\}$$

$$\sum_{\substack{\lambda_1 \lambda_2 \\ \rho_1 \rho_2}} \eta^\dagger_{\lambda_1} \eta^\dagger_{\lambda_2} \zeta^\dagger_{\rho_1} \zeta^\dagger_{\rho_2}\, \eta_{\lambda_2} \eta_{\lambda_1} \zeta_{\rho_2} \zeta_{\rho_1}$$

$\delta_{\lambda_1 \lambda_2}$

$\delta_{\lambda_2 \lambda_1}$

$\delta_{\rho_1 \rho_2}\ \delta_{\rho_2 \rho_1}$

spin isospin line $= 2 \cdot 2 = 4$

Exercise

$$V_{exchange} = \frac{-k_F^3 A}{48\pi^2}\left\{4\, a_W \int V_0 \left[\frac{3 j_1(k_F r)}{k_F r}\right]^2 d^3z + 4\, a_M \int V_0\, d^3z\right\}$$

Final result

$$\frac{E_0+E_1}{A} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m} + \frac{k_F^3}{12\pi^2}\left\{(4a_W - a_M)\int V_0\, d^3z + (4a_M - a_W)\int\left[\frac{3j_1(k_F r)}{k_F r}\right]^2 V_0\, d^3z\right\}$$

Now consider the limit k_F □ ∞ ;
i.e., the particle density □ ∞.

E_1/A is the dominant term (k_F^3 vs k_F^2).

If $4a_W - a_M > 0$, then E_1 □ $-\infty$ as k_F □ ∞;
because the term proportional to $\int V_0(r) d^3z$ is dominant and negative.

(Remember, we're assuming $V_0(r)$ is attractions (negative) and nonsingular.

Recall the "Serber force": $a_M = a_W$.
From low-energy nucleon-nucleon scattering experiments, $a_M \approx a_W$.

So, the nuclear matter will collapse to infinite density.

"Saturation" does not occur for a nonsingular Serber force; so the next step is to include the hard core potential (Section 41).

Define a single-particle potential, U(**k**)

$$U_{\lambda\rho}(\vec{k}) = \sum_{\vec{k}'\lambda'\rho'}^{k_F} \left\{ \langle k k' | V | k k' \rangle - \langle k k' | V | k' k \rangle \right\}$$

Note that $E_1 = \frac{1}{2} \sum_{\vec{k}\lambda\rho} U_{\lambda\rho}(\vec{k})$

$$U(\vec{k}) = \frac{k_F^3}{6\pi^2} \left\{ (4a_W - a_M) \int V_0 \, d^3z + (4a_M - a_W) \int j_0(k_F r) \frac{3 j_1(k_F r)}{k_F r} V_0 \, d^3z \right\}$$

F&W provide the *interpretation* of U(k), and discuss the limits of small k and large k .

Interpretation:

- a kind of single particle energy
- It's the Hartree-Fock potential.
- It's the first order contribution to the self energy.

Sketch:

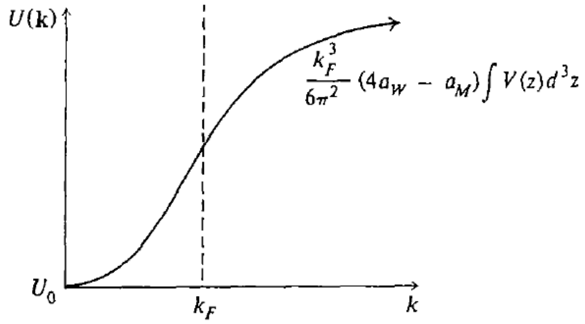


Fig. 40.1 Sketch of the single-particle potential $U(\mathbf{k})$ in Eq. (40.19). See also Eq. (40.23).

The effective mass formula,

$m^* = m \;/\; [\, 1 + U''(0) \,]$.