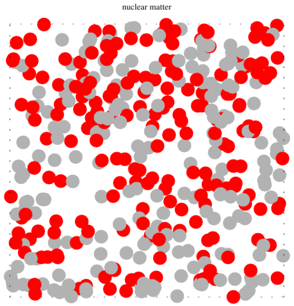
Chapter 11: Nuclear Matter



So far we have assumed a nonsingular nucleon-nucleon force of a Serber exchange nature and have calculated the ground-state energy shift to lowest order in the interaction. This result is very powerful since it gives a variational bound on the true ground-state energy and shows that the assembly is unstable against collapse with such a Serber force. We are now faced with two problems. First, how do we explain nuclear saturation? The answer is that the potential has been assumed to be nonsingular, whereas nuclear forces are actually singular. As seen in Sec. 38, there is evidence for a strong repulsion at short distances, which must be included in the calculation. The second problem is to understand the success of the independent-particle model of the nucleus. It is clear that the singular nuclear forces introduce important correlations. Nevertheless, the numerous triumphs of the single-particle shell model of the nucleus and the accurate description of scattering through a single-particle optical potential show that the independent-particle model frequently represents an excellent starting approximation in nuclear physics. In Sec. 41 we attempt to answer these questions with the *independent-pair approximation*, in which two-body correlations are treated in detail.

41. Independent pair approximation (Brueckner's theory)

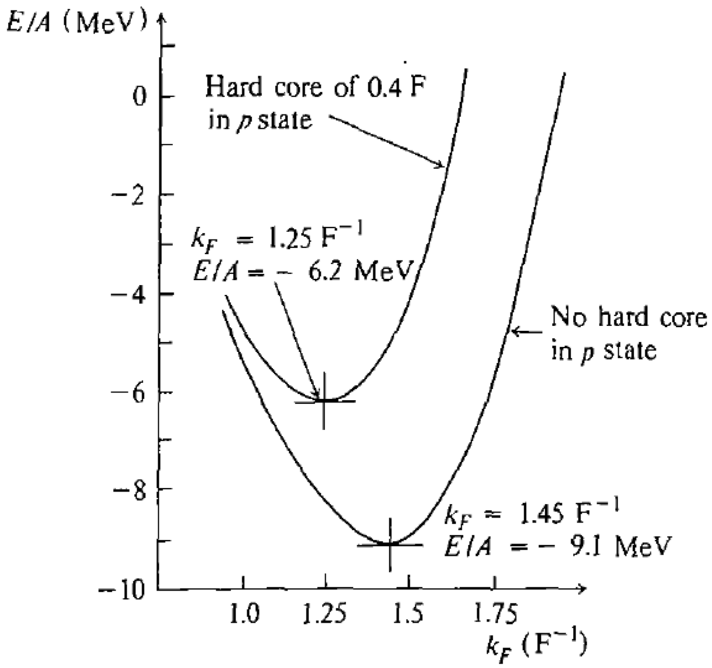


Fig. 41.7 The energy per particle in nuclear matter as a function of k_F computed from Eqs. (41.57), (41.58), (41.67), and (41.72) for the two-body potential of Eqs. (41.39) to (41.42). The results are shown both with and without a hard core in the p state. (The authors wish to thank E. Moniz for preparing this figure.)

41. Independent pair approximation (Brueckner’s theory)

The idea is:

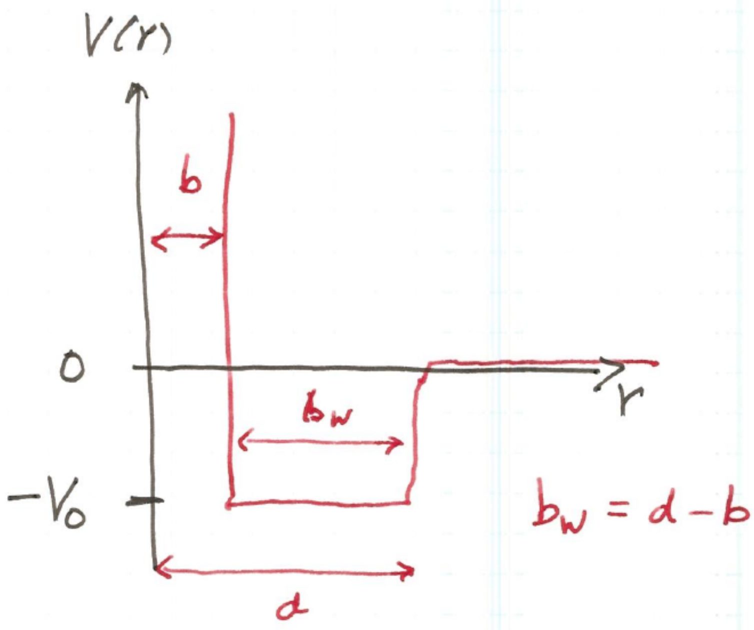
Consider two particles in the nuclear matter (a “pair”).

Calculate their interaction energy, neglecting the other particles.

Sum over pairs to get the ground state energy.

Introduce a “realistic 2-particle potential”
(actually very simplified!)

Figure 41.5
Hard-core square-well potential



$$V(r) = \begin{cases} \infty & \text{if } r < b \\ -V_0 \frac{1}{2}(1+P_M) & \text{if } b < r < d \\ 0 & \text{if } r > d \end{cases}$$

Three parameters

b, d, V_0

- There is a weakly bound bound state
  $\therefore \quad V_0 \gtrsim \dfrac{\hbar^2 \pi^2}{4m\, b_W^2}$
- Hard core radius $\Rightarrow$ $b \approx 0.4$ fm.
- Choose parameter values
  $b = 0.4$ fm
  $b_W = 1.9$ fm ; or $d = 2.3$ fm
  $V_0 = 28$ MeV

Recall $k_F = 1.42\ \mathrm{fm}^{-1}$; $\therefore$ $k_F d = 3.27$

The Bethe-Goldstone equation

For a pair of interacting particles in the nuclear matter, with total momentum **P** and relative momentum **k,** the relative wave function is $\psi_{\mathbf{Pk}}(\mathbf{x})$
(suppressing spin-isospin indices) is

$$(1) \quad \psi_{Pk}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + \int_{\Gamma} \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \frac{\int d^3y\, e^{-i\vec{q}\cdot\vec{y}}\, V(\vec{y})\, \psi_{Pk}(\vec{y})}{E_{Pk} - \epsilon_{q_1} - \epsilon_{q_2}}$$

where

$$\epsilon_{\vec{p}} = \epsilon^{0}_{\vec{p}} + U(\vec{p}) \quad ; \quad \epsilon^{0}_{p} = \frac{\hbar^2 p^2}{2m}$$

$$\Gamma = \{\, \vec{q} \;;\; |\tfrac{1}{2}\vec{P} + \vec{q}| > k_F \text{ and } |\tfrac{1}{2}\vec{P} - \vec{q}| > k_F \,\}$$

$$F = \{\, \vec{k} \;;\; |\tfrac{1}{2}\vec{P} + \vec{k}| < k_F \text{ and } |\tfrac{1}{2}\vec{P} - \vec{k}| < k_F \,\}$$

$$\vec{q}_1 = \tfrac{1}{2}\vec{P} + \vec{q} \quad \text{and} \quad \vec{q}_2 = \tfrac{1}{2}\vec{P} - \vec{q}$$

$$(2) \quad E_{Pk} - \epsilon_{k_1} - \epsilon_{k_2} = \Delta\epsilon_{Pk} \qquad \left\{ \begin{matrix} \vec{k}_1 \\ \vec{k}_2 \end{matrix} \right. = \vec{P} \pm \tfrac{1}{2}\vec{k}$$

$$= \frac{1}{V} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, V(x)\, \psi_{\vec{P},\vec{k}}(\vec{x})$$

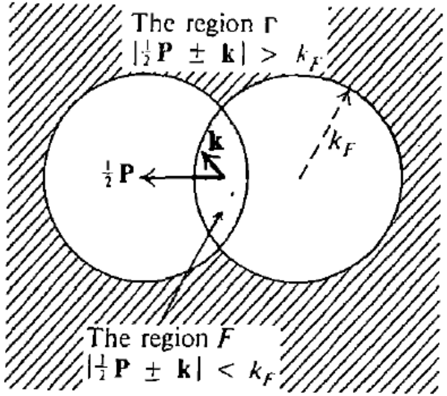


Fig. 41.1 Momentum regions in the Bethe-Goldstone equations.

$$(1) \quad \psi_{Pk}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + \int_{\Gamma} \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \frac{\int d^3y\, e^{-i\vec{q}\cdot\vec{y}} V(\vec{y})\, \psi_{Pk}(\vec{y})}{E_{Pk} - \epsilon_{q_1} - \epsilon_{q_2}}$$

$$\varepsilon_p = \varepsilon^0_p + U(p)$$

Recall scattering theory

$$(H_0 + V)\psi = E\psi$$

Write $\psi = \phi + \hat{\psi}$ where $\phi =$ plane wave

where $H_0\phi = E\phi$

Then

$$E\phi + H_0\hat{\psi} + V\psi = E\phi + E\hat{\psi}$$

$$(E - H_0)\hat{\psi} = V\psi$$

$$\hat{\psi} = (E - H_0)^{-1} V\psi$$

$$\psi = \phi + (E - H_0)^{-1} V\psi$$

$$(2) \quad E_{Pk} - \epsilon_{k_1} - \epsilon_{k_2} = \Delta\epsilon_{Pk} \qquad \left\{ \begin{array}{l} \vec{k}_1 \\ \vec{k}_2 \end{array} \right. = \vec{P} \pm \tfrac{1}{2}\vec{k}$$

$$= \frac{1}{V} \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, V(x)\, \psi_{\vec{P},\vec{k}}(\vec{x})$$

So the result is a system of equations for $\psi_{Pk}(\mathbf{x})$ and $U(\mathbf{k})$.

I.e., it is a *self-consistent* theory:

Eq. (1) ←→ Eq. (2)

Step 1:

Consider a purely attractive potential.

⇒A purely attractive potential has almost no effect on the 2-particle wave function.

⇒The long range attraction of $V_0(r)$ scarcely affects the 2-particle wave function.

Step 2:

Consider a purely hard-core potential.

⇒"healing distance" ;

⇒"interparticle distance" is larger ;

"These observations also provide a simple qualitative basis for the independent-particle model of nuclear matter."
The Pauli exclusion principle restricts the effect of the hard core to short distances; the repulsion does not produce long range correlations.

Calculations

$$\Delta \epsilon_{Pk} = \frac{1}{V} \int e^{-i\vec{k}\cdot\vec{x}} \left(V_{att.} + V_{rep.} \right) \psi_{Pk} \, d^3x$$

$$U(\vec{k}\lambda_1 \rho_1) = \sum_{\vec{k}_2, \lambda_2, \rho_2}^{k_F} \Delta \epsilon_{P,k}$$

$$\Delta E = \frac{1}{2} \sum_{k_1 \lambda_1 \rho_1}^{k_F} \sum_{k_2 \lambda_2 \rho_2}^{k_F} \Delta \epsilon_{P,k}$$

For the reasonable approximation,

(i) replace $\psi \approx \psi_{hardcore}$

(ii) replace $\psi_{hardcore} \approx e^{i\vec{k}\cdot\vec{x}}$

in the attractive well $(b < r < b + b_w)$

$$\therefore \Delta \epsilon_{Pk} = \Delta \epsilon_{Pk}^{(att.)} + \Delta \epsilon_{Pk}^{(rep.)}$$

$\uparrow$ already calculated in Section 40.

Results

Results

$$U^{att.}(\vec{k}) = \frac{-V_0 k_F^3}{3\pi} \left[d^3 - b^3 + \frac{9}{k_F} \int_b^d j_0(kr) \, j_1(k_F r) \, r \, dr \right]$$

$$\frac{E^{att.}}{A} = \frac{-V_0}{6\pi} \left\{ \rho^3 + 9 f(\rho) \right\} \Big|_{\rho = k_F b}^{k_F d}$$

where $f(\rho) = Si(2\rho) + \frac{\cos 2\rho - 3}{2\rho} + \frac{\sin 2\rho}{\rho^2} + \frac{\cos 2\rho - 1}{2\rho^3}$

$$\frac{E^{(hardcore)}}{A} = \frac{\hbar^2 k_F^2}{2m^*} \left(\alpha c + \beta c^2 + O(c^3) \right)$$

where

$$\alpha = \frac{2}{\pi} \iint_{(F)} \frac{d^3K}{4\pi/3} \frac{d^3P}{4\pi/3} = \frac{2}{\pi}$$

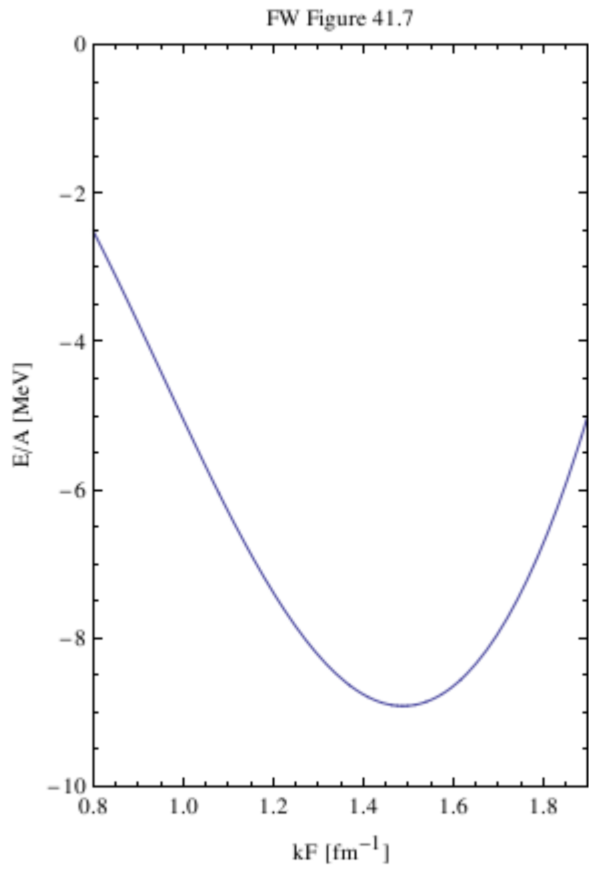
$$\beta = \frac{2}{\pi^2} \iint_{(F)} \frac{d^3K}{4\pi/3} \frac{d^3P}{4\pi/3} \left[1 + \frac{P}{2} + K \ln\left(\frac{1 + P/2 - K}{1 + P/2 + K} \right) + - \frac{K^2 + (P/2)^2 - 1}{P} \ln \frac{(1 + P/2)^2 - K^2}{1 - (P/2)^2 - K^2} \right]$$

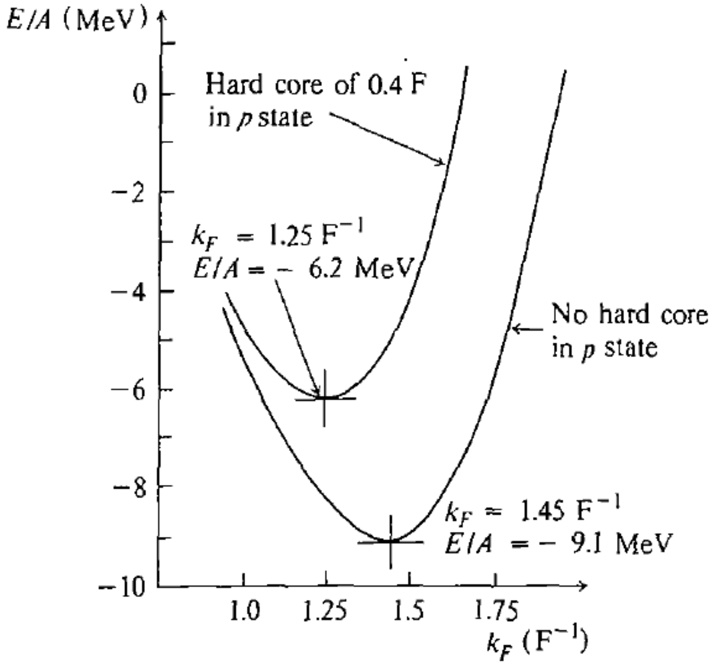
$$= \frac{12}{35\pi^2} \left(11 - 2 \ln 2 \right)$$

$$c = k_F b = 0.57$$

and don't forget

$$\frac{E^{(free)}}{A} = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m}$$





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