How is the Dirac equation consistent with special relativity? (Sections 3.2 and 3.4)

$$(i\ \gamma.\partial - m)\ \psi(x) = 0 \qquad (1)$$

Consider two inertial frames,

$x^\mu = \{ x^0, x^1, x^2, x^3 \}$ ref.frame $\mathcal{F}$

$x'^{\mu} = \{ x'^0, x'^1, x'^2, x'^3 \}$ ref.frame $\mathcal{F}'$

$x'^{\mu} = \Lambda^\mu{}_\nu\, x^\nu$ where $\Lambda^\mu{}_\nu$ = Lorentz transformation matrix

Suppose $\psi(x)$ is a solution of the Dirac equation in the unprimed coordinates. *What is the solution in the primed coordinates?*

I.e., we want

$$(i\ \gamma'.\partial' - m)\ \psi'(x') = 0 \qquad (2)$$

Question: What is γ'^{μ} ?
Answer: $\gamma'^{\mu} = \gamma^\mu$.
Proof: Gamma matrices are the same in all inertial frames; e.g.,

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} .$$

Question: What is ∂'_μ ?
Answer: $\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu$
Proof: By the chain rule,

$$\frac{\partial Q}{\partial x'^\mu} = \frac{\partial Q}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} \quad \text{where } x^\nu = (\Lambda^{-1})^\nu{}_\rho\, x'^\rho$$

$$\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu$$

Require $(i\ \gamma'.\partial' - m)\, \psi'(x') = 0$ (2) .

Question: What is $\psi'(x')$?

Answer:

$$\psi'(x') = S(\Lambda)\, \psi(x)$$

[PS notation: $\Lambda_{1/2} = S(\Lambda)$]

Here $S(\Lambda)$ is the 4 x 4 matrix, such that Eq. (2) is satisfied.

Theorem 1.

The matrix $S(\Lambda)$ satisfies

$$S^{-1}\, \gamma^{\mu}\, S = \Lambda^{\mu}{}_{\nu}\ \gamma^{\nu} \quad .$$

Theorem 2. $S(\Lambda) = \exp[- (i/2)\, \omega_{\mu\nu} S^{\mu\nu}]$

where $S^{\mu\nu} = i/4\, [\gamma^{\mu},\gamma^{\nu}]$

and $\omega_{\mu\nu}$ is antisymmetric w.r.t. $\mu\nu$ exchange.

Proof of Theorem 1.

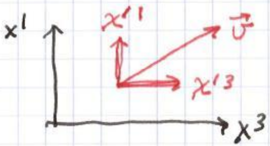
$$\begin{aligned}
&(i\ \gamma'.\partial' - m)\, \psi'(x') \\
&\quad = (i\, \gamma^{\mu}\ (\Lambda^{-1})^{\nu}{}_{\mu}\, \partial_{\nu} - m)\, S(\Lambda)\, \psi(x) \\
&\quad = S\, S^{-1} (i\, \gamma^{\mu}\ (\Lambda^{-1})^{\nu}{}_{\mu}\, \partial_{\nu} - m)\, S\, \psi(x) \\
&\quad = S\, (i\, S^{-1} \gamma^{\mu}\, S\, (\Lambda^{-1})^{\nu}{}_{\mu}\, \partial_{\nu} - m)\, \psi(x) \\
&\quad = S\, (i\, \Lambda^{\mu}{}_{\rho}\ \gamma^{\rho}\, (\Lambda^{-1})^{\nu}{}_{\mu}\, \partial_{\nu} - m)\, \psi(x) \\
&\quad = S\, (i\, \gamma^{\rho}\, \partial_{\rho} - m)\, \psi(x) \\
&\quad = 0 \, .
\end{aligned}$$

Q.E.D. (assuming S exists)

Consider a boost by velocity $\vec{v} = \beta\hat{n}$.

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -n^1\beta\gamma & -n^2\beta\gamma & -n^3\beta\gamma \\ -n^1\beta\gamma & 1+n_1^2\rho & n^1n^2\rho & n^1n^3\rho \\ -n^2\beta\gamma & n^2n^1\rho & 1+n_2^2\rho & n^2n^3\rho \\ -n^3\beta\gamma & n^3n^1\rho & n^3n^2\rho & 1+n_3^2\rho \end{bmatrix}$$

where $\rho = \gamma - 1$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.



Theorem 2.

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

Guess

$$\omega_{\rho\sigma} = \begin{bmatrix} 0 & -\omega n^1 & -\omega n^2 & -\omega n^3 \\ \omega n_1 & 0 & 0 & 0 \\ \omega n^2 & 0 & 0 & 0 \\ \omega n^3 & 0 & 0 & 0 \end{bmatrix}$$

$$\omega \cdot S = -\omega n^i S^{0i} \quad \times 2$$

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{i}{4}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix} - \begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix}0&\sigma^i\\\sigma^i&0\end{pmatrix}$$

$$\omega \cdot S = -i\omega\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix}$$

$$\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix}\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$S(\Lambda) = e^{-\frac{\omega}{2}\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix}} = \begin{pmatrix}\cosh\frac{\omega}{2} & -\sigma_n\sinh\frac{\omega}{2}\\ -\sigma_n\sinh\frac{\omega}{2} & \cosh\frac{\omega}{2}\end{pmatrix}$$

$$= \begin{pmatrix}c_2 & -\sigma_n s_2\\ -\sigma_n s_2 & c_2\end{pmatrix}$$

$$S^{-1}(\Lambda) = \begin{pmatrix}c_2 & \sigma_n s_2\\ \sigma_n s_2 & c_2\end{pmatrix}$$

Check $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$x'^0 = \gamma x^0 - \beta\gamma\,\hat{n}\cdot\vec{x}$$

$$= \gamma x^0 - \gamma\,\vec{v}\cdot\vec{x} \quad ✓$$

$$x'^n = n^i\Lambda^i{}_\nu x^\nu$$

$$= n^i(-n^i\beta\gamma)x^0 + n^i\Lambda^i{}_j x^j$$

$$= -\beta\gamma x^0 + n^i[\delta^i_j + n^i n^j\rho]x^j$$

$$= -\beta\gamma x^0 + (1+\rho)\,\hat{n}\cdot\vec{x}$$

$$= -\beta\gamma x^0 + \gamma x^n \quad ✓$$

$$\hat{n}\times\vec{x}' = \hat{n}\times(\hat{e}_i \underbrace{\Lambda^i{}_\nu}_{\text{only } \delta^i_j \text{ is nonzero}} x^\nu) = \hat{n}\times\vec{x} \quad ✓$$

Theorem 2. **To prove: $S^{-1}\gamma^{\mu} S = \Lambda^{\mu}{}_{\nu}\ \gamma^{\nu}$**

$\mu = 0$

$$\begin{pmatrix} c_2 & \sigma_n s_2 \\ \sigma_n s_2 & c_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_2 & -\sigma_n s_2 \\ -\sigma_n s_2 & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_2^2 + s_2^2 & -\sigma_n 2 c_2 s_2 \\ \sigma_n 2 c_2 s_2 & -c_2^2 - s_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} C & -\sigma_n S \\ \sigma_n S & -C \end{pmatrix} \qquad \begin{matrix} C = \cosh\omega \\ S = \sinh\omega \end{matrix}$$

$$= \Lambda^0{}_\nu \gamma^\nu = \Lambda^0{}_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Lambda^0{}_i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda^0{}_0 & \sigma^i \Lambda^0{}_i \\ -\sigma^i \Lambda^0{}_i & -\Lambda^0{}_0 \end{pmatrix}$$

$\therefore\ \Lambda^0{}_0 = \cosh\omega = \gamma$

& $\Lambda^0{}_i = -n^i S = -n^i \sinh\omega$

$= -n^i \beta\gamma$

$\gamma^2 - \beta^2\gamma^2 = 1$ so $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\mu = i$

$$\begin{pmatrix} c_2 & \sigma_n c_2 \\ \sigma_n c_2 & c_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} c_2 & -\sigma_n s_2 \\ -\sigma_n s_2 & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -n^i S & \sigma^i + 2n^i \sigma_n s_2^2 \\ -\sigma^i - 2n^i \sigma_n s_2^2 & n^i S \end{pmatrix}$$

$$= \Lambda^i{}_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Lambda^i{}_j \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

Exercise :

$\Lambda^i{}_0 = -n^i S \qquad (S = \sinh\omega)$ $(\beta\gamma)$

$\Lambda^i{}_j = \delta_{ij} + 2 n^i n^j s_2^2$

$= \delta_{ij} + n^i n^j (C - 1)$ $(\gamma - 1)$

Q.E.D.

Example.

Consider a boost in the z direction.

The Lorentz transformation matrix:

$$
\begin{aligned}
x'^0 &= [\ \gamma \quad 0 \quad 0 \quad -\beta\gamma\ ]\ x^0 \\
x'^1 &= [\ 0 \quad 1 \quad 0 \quad 0\ ]\ x^1 \\
x'^2 &= [\ 0 \quad 0 \quad 1 \quad 0\ ]\ x^2 \\
x'^3 &= [\ -\beta\gamma \quad 0 \quad 0 \quad \gamma\ ]\ x^3
\end{aligned}
$$

$$\Lambda^{\mu}{}_{\nu}$$

What is S(Λ) ?

$$S(\Lambda) = \exp\{-\omega/2 \begin{bmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{bmatrix}\}$$

$$S(\Lambda) = \begin{bmatrix} \cosh(\omega/2) & -\sigma^3\sinh(\omega/2) \\ -\sigma^3\sinh(\omega/2) & \cosh(\omega/2) \end{bmatrix}$$

$$S(\Lambda) = \begin{bmatrix} \cosh(\omega/2) & -\sigma^3\sinh(\omega/2) \\ -\sigma^3\sinh(\omega/2) & \cosh(\omega/2) \end{bmatrix}$$

The Dirac spinors for a particle at rest are

$$u'(0,1) = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \text{and} \quad u'(0,2) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

(Frame $\mathcal{F}'$ is the rest frame of the particle)

Therefore the Dirac spinors for a particle with momentum p = (0, 0, p^3) are

$$u(p,1) = S^{-1}(\Lambda)\, u'(0,1)$$

$$= \begin{pmatrix} c_2 & +\sigma^3 s_2 \\ +\sigma^3 s_2 & c_2 \end{pmatrix} \sqrt{2m} \begin{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} \end{pmatrix}$$

$$= \sqrt{2m}\begin{pmatrix} c_2\\0\\s_2\\0 \end{pmatrix} \quad \text{where}$$

$$\boxed{\begin{aligned} c_2 &= \sqrt{\tfrac{1}{2}(C+1)} = \sqrt{\tfrac{1}{2}(\gamma+1)} \\ &= \sqrt{\frac{E+m}{2m}} = \frac{E+m}{\sqrt{2m(E+m)}} \\ \text{and} \quad s_2 &= \sqrt{\tfrac{1}{2}(C-1)} = \sqrt{\tfrac{1}{2}(\gamma-1)} \\ &= \sqrt{\frac{E-m}{2m}} = \frac{p^3}{\sqrt{2m(E+m)}} \end{aligned}}$$

$$= \frac{1}{\sqrt{E+m}}\begin{pmatrix} E+m\\0\\p^3\\0 \end{pmatrix}$$

and sim. $u(p,2) = \dfrac{1}{\sqrt{E+m}}\begin{pmatrix} 0\\E+m\\0\\-p^3 \end{pmatrix}$

Dirac Field Bilinears

$\bar{\psi}\ \psi$ is a scalar

$\bar{\psi}\ \gamma^{\mu}\ \psi$ is a vector

$\bar{\psi}\ \gamma^{\mu}\ \gamma^{\nu}\ \psi$ is a tensor

$\bar{\psi}\ \gamma^{5}\ \psi$ is a pseudo-scalar

$\bar{\psi}\ \gamma^{\mu}\ \gamma^{5}\ \psi$ is a pseudo-vector

Proof

$$\bar{\psi}'\,\psi' = \psi'^{\dagger}\gamma^0\psi'$$

$$= \psi^{\dagger}\,S(\Lambda)^{\dagger}\,\gamma^0\,S(\Lambda)\,\psi$$

$$= \psi^{\dagger}\gamma^0\,\underline{\gamma^0\,S(\Lambda)^{\dagger}\,\gamma^0}\,S(\Lambda)\,\psi$$

where

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$\gamma^0 S(\Lambda)^{\dagger}\gamma^0 = \gamma^0\, e^{\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^{\dagger}}\,\gamma^0$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$ because $\gamma^0 (S^{\mu\nu})^{\dagger}\gamma^0 = S^{\mu\nu}$

$$\gamma^0 S(\Lambda)^{\dagger}\gamma^0 S(\Lambda) = e^{\frac{i}{2}\omega\cdot S}\, e^{-\frac{i}{2}\omega\cdot S} = 1$$

$$\bar{\psi}'\,\psi' = \bar{\psi}\,\psi$$ Q.E.D.

Etc, similarly.

Homework Assignment #3