Chapter 4 =

We will derive the methods for calculating transition matrix elements (Feynman diagrams) and reaction cross sections.

But first let's do an example.

This will illustrate:

- What are the computational techniques?
- What are the theoretical issues to understand?

Here is the example

Pion proton scattering in a RQFT for the π p interaction, with a pseudoscalar coupling

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}_{int}$$

$$\mathcal{L}_\psi = \overline{\Psi}\,(i\not{\partial} - M)\,\psi$$

spin and isospin

$$\mathcal{L}_\phi = \frac{1}{2}\partial^\mu \phi_a \partial_\mu \phi_a - \frac{m^2}{2}\phi_a \phi_a$$

$$\pi^0 \text{ field} = \phi_3$$

$$\pi^\pm \text{ field} = \frac{1}{\sqrt{2}}(\phi_1 \pm i\phi_2)$$

$$\mathcal{L}_{int} = g\,\overline{\Psi}\,\tau_a \gamma_5 \Psi\,\phi_a$$

isospin Pauli matrices

Recall from PHY 855

$$\langle f |\, T \exp(-i) \int_{-T}^{T} H_1 \, dt \,| i \rangle$$

= probability amplitude for evolution from $|i\rangle$ to $|f\rangle$

= the probability amplitude for evolution from initial state |i> at time -T to final state |f> at time +T; take the limit T → infinity.

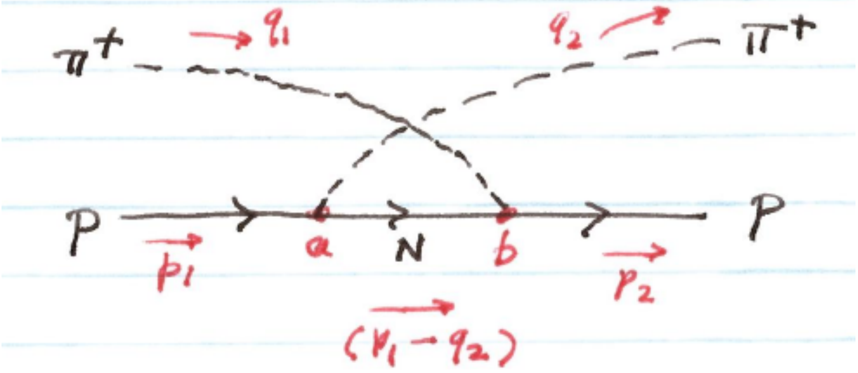
The interaction Hamiltonian is

$$H_1 = -\int \mathcal{L}_{int} \, d^3x$$

For the lowest order calculation, the relevant term in the perturbation expansion is $L_{int}^{\,2}$ ...

$$[g \bar{\Psi} \tau_a \gamma_5 \Psi \phi_a](x) \; [g \bar{\Psi} \tau_b \gamma_5 \Psi \phi_b](y)$$

Consider π^+ p scattering.
There is one Feynman diagram



The scattering amplitude

$$\langle \pi_2 P_2 | \left[g \bar{\Psi} \tau_a \gamma_5 \Psi \phi_a \right](x) \left[g \bar{\Psi} \tau_b \gamma_5 \Psi \phi_b \right](y) | \pi_1 P_1 \rangle \int d^4x \int d^4y$$

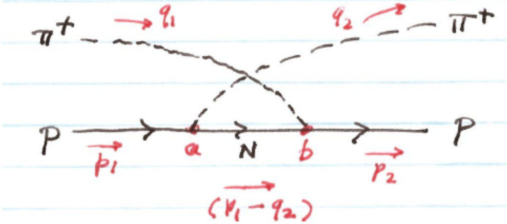
creates $P(p_2)$ — creates $\pi^+(q_2)$ — annihilates $P(p_1)$ — annihilates $\pi^+(q_1)$

Wick's theorem ⇒ contraction $\Psi(x)\bar{\Psi}(y) = S_F(x-y)$

Transformation to momentum space

$$\mathcal{M} = g^2 \, \bar{u}(p_2) \xi_2^\dagger \tau_b \gamma_5 S_F(p1-q2) \gamma_5 \tau_a \xi_1 u(p_1)$$

$$\frac{1}{\sqrt{2}}(\delta_{b1} + i\delta_{b2}) \frac{1}{\sqrt{2}}(\delta_{a1} - i\delta_{a2})$$



(1) Isospin

$$\xi_2^\dagger \underbrace{\tau_b \tau_a}_{} \xi_1 \frac{1}{2}(\delta_{b1} + i\delta_{b2})(\delta_{a1} - i\delta_{a2})$$

$\leftrightarrow \delta_{ba} + i\epsilon_{bac}\tau_c$ (Pauli matrices)

$$(1\ 0)\left[\delta_{ba} + i\epsilon_{bac}\tau_c\right]\begin{pmatrix}1\\0\end{pmatrix} = \delta_{ba} + i\epsilon_{ba3}$$

$$= \frac{1}{2}\left[(\delta_{b1} + i\delta_{b2})(\delta_{b1} - i\delta_{b2}) + i\epsilon_{123}(1)(-i) + i\epsilon_{213}(i)(1)\right]$$

$$= \frac{1}{2}\left[1 + 1 + 1 + 1\right] = 2$$

Comment $\xi_2^\dagger \tau_b \xi_j \xi_j^\dagger \tau_a \xi_1$

1, 2 : proton states $\begin{pmatrix}1\\0\end{pmatrix}$

a, b = 1 or 2 ⇒ $\xi_j = \begin{pmatrix}0\\1\end{pmatrix}$ neutron

(2) Spin

$$\bar{u}_2 \gamma_5 \frac{i(\not{p}_1 - \not{q}_2 + M)}{(p_1 - q_2)^2 - M^2} \gamma_5 u_1$$

$$= \frac{i}{u - M^2} \bar{u}_2 (-\not{p}_1 + \not{q}_2 + M) u_1$$

$u = (p_1 - q_2)^2$

$$= \frac{i}{u - M^2} \bar{u}_2 \not{q}_2 u_1$$

$$\mathcal{M} = \frac{2g^2}{u - M^2} \bar{u}(p_2) \not{q}_2 u(p_1)$$

(3) Amplitude squared

$$\mathcal{M} = \frac{2g^2}{u - M^2} \bar{u}(p_2) \not{q}_2 u(p_1)$$

$\hookrightarrow \bar{u}_2 = u_2^\dagger \gamma^0$

$$|\mathcal{M}|^2 = \frac{4g^4}{(u - M^2)^2} u_2^\dagger \gamma^0 \not{q}_2 u_1 \; u_1^\dagger \not{q}_2^\dagger (\gamma^0)^\dagger u_2$$

$(ABC\ldots)^\dagger = (\ldots C^\dagger B^\dagger A^\dagger)$

$(\gamma^0)^\dagger = \gamma^0 \; ; \; \gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$

$\gamma^0 (\not{q}_2)^\dagger \gamma^0 = \not{q}_2$

$$|\mathcal{M}|^2 = \frac{4g^4}{(u - M^2)^2} (\bar{u}_2 \not{q}_2 u_1)(\bar{u}_1 \not{q}_2 u_2)$$

$$= \frac{4g^4}{(u - M^2)^2} \mathrm{Tr}\, \not{q}_2 \, u_1 \bar{u}_1 \not{q}_2 \, u_2 \bar{u}_2$$

$\sum_s u_1 \bar{u}_1 = \not{p}_1 + M$

(4) Traces for unpolarized scattering

$$\overline{|\mathcal{M}|^2} = \frac{4g^4}{(u - M^2)^2} \frac{1}{2} \mathrm{Tr}\, \not{q}_2 (\not{p}_1 + M) \not{q}_2 (\not{p}_2 + M)$$

$$= \frac{2g^4}{(u - M^2)^2} \{ 4\, q_2 \cdot p_1 \, q_2 \cdot p_2 - 4 m^2 \, p_1 \cdot p_2 + 4\, q_2 \cdot p_2 \, q_2 \cdot p_1 + M^2 \, 4 m^2 \}$$

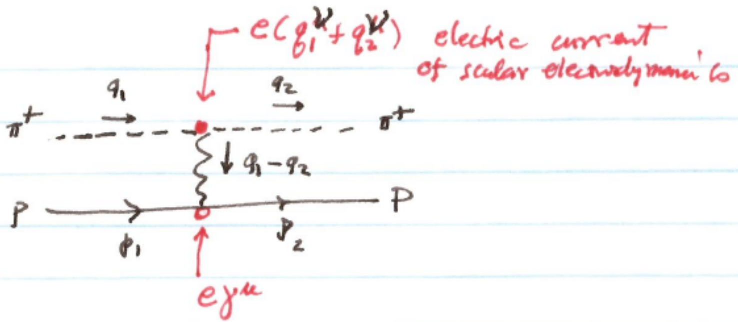
$$= \frac{8g^4}{(u - M^2)^2} \{ 2\, q_2 \cdot p_1 \, q_2 \cdot p_2 + m^2 (M^2 - p_1 \cdot p_2) \}$$

(5) Mandelstam variables (see the handout)

$$\overline{|\mathcal{M}|^2} = \frac{4g^4}{(u-M^2)^2}\left\{(s-m^2-M^2)^2 + (s-M^2)t\right\}$$

But that is not good enough.

We should also include the electromagnetic interaction. The two interactions *interfere*. We must add the amplitudes and then square the combined amplitude.



$$\mathcal{M}_e = \bar{u}(p_2)\, e\gamma^\mu u(p_1)\, e(q_1^\nu + q_2^\nu)\, D_{F\mu\nu}(q_1 - q_2)$$

where the photon propagator is

$$D_{F\mu\nu}(Q) = \frac{g_{\mu\nu}}{Q^2}$$

$$\mathcal{M}_e = \frac{e^2}{t}\,\bar{u}_2(\not{q}_1 + \not{q}_2)u_1$$

$$\mathcal{M} + \mathcal{M}_e = \bar{u}_2\left[\frac{2g^2\not{q}_2}{u-M^2} + \frac{e^2}{t}(\not{q}_1 + \not{q}_2)\right]u_1$$

$$= \bar{u}_2(\alpha\not{q}_2 + \beta\not{q}_1)u_1$$

$$\overline{|\mathcal{M} + \mathcal{M}_e|^2} = \frac{1}{2}\mathrm{Tr}\left[(\alpha\not{q}_2 + \beta\not{q}_1)(\not{p}_1 + M)(\alpha\not{q}_2 + \beta\not{q}_1)(\not{p}_2 + M)\right]$$

Do the traces.

Substitute Mandelstam variables.

Simplify.

After some pages of algebra

$$\overline{|\mathcal{M}+\mathcal{M}_e|^2} = (\alpha+\beta)^2 \left\{(s-m^2-M^2)^2 + (s-M^2)t\right\}$$

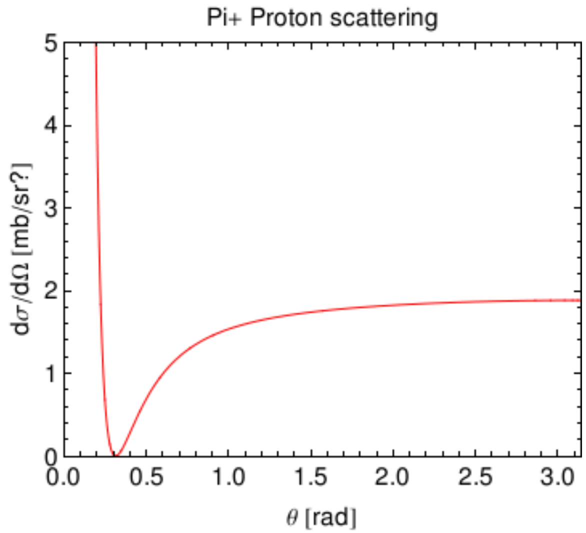
$$\alpha = \frac{2g^2}{u-M^2} + \frac{e^2}{t} \;; \quad \beta = \frac{e^2}{t} \,.$$

Calculate the cross section

See the handout.

Let's use the center of mass frame of reference. Then we can apply the special case formula (homework problem 6).

$$d\sigma / d\Omega_3 = |\mathcal{M}|^2 / (64\, \pi^2\, s) \,.$$

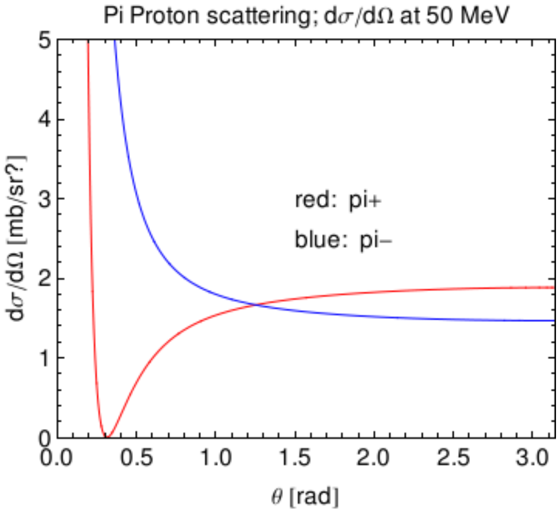


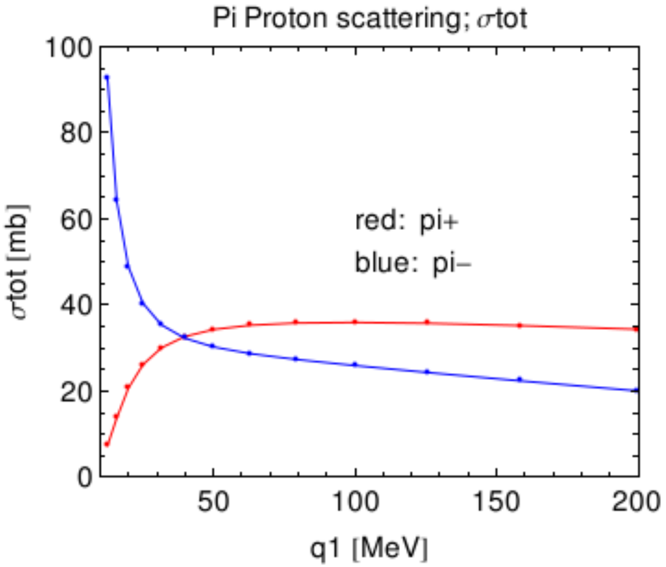
Pi-minus P scattering is slightly different.

Pi-minus P

Pi-plus scattering

Comparing Pi+ P and Pi- P scattering.





Some data on Pi-plus Proton scattering

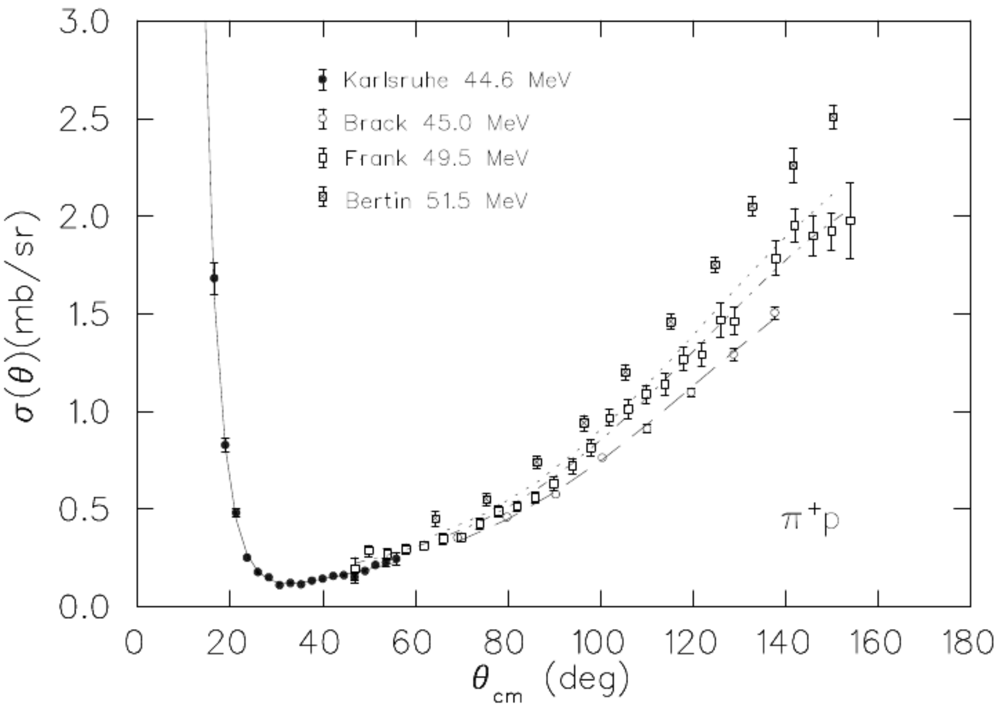


FIG. 1. Cross sections from π^+ proton scattering around 50 MeV. The Bertin data would agree with the prediction given by the dotted curve if they were consistent with the other data sets. The solid, long dash and dash-dot curves come from our fit and correspond to the Karlsruhe, Brack and Frank data sets.

The Delta Resonance Δ(1232);

root-S = 1232 MeV ; q1 = 220 MeV.