CHAPTER 5 : EXAMPLES IN QUANTUM ELECTRODYNAMICS

These are the processes that are calculated in chapter 5:

∎ $e^+ e^- \rightarrow \mu^+ \mu^-$
This is a basic example in QED. Also important because of the similar process $e^+ e^- \rightarrow q \bar{q}$, which proved the quark model and QCD by describing $qqbar \rightarrow$ hadrons; PETRA (DESY) and PEP (SLAC) accelerator experiments.

∎ $e^- + \mu^- \rightarrow e^- + \mu^-$
An example of crossing symmetry. Is scattering by a muon realistic? No, but the muon could be the projectile. also important because of the similar process $e^- + q \rightarrow e^- + q$, which occurs in electron-proton deep-inelastic scattering (ep DIS); SLAC and HERA (DESY) experiments.

∎ $\gamma e^- \rightarrow \gamma e^-$
Compton scattering.

∎ $e^+ e^- \rightarrow \gamma \gamma$
Pair annihilation into photons.

Other examples

∎ $e^- e^- \rightarrow e^- e^-$
Moller scattering. This is interesting as an example of identical particles.

∎ $e^+ e^- \rightarrow e^+ e^-$
Bhabha scattering. This process is used to determine the luminosity of $e^+ e^-$ collisions. Provides the best upper limit on the radius of the electron.
THE FEYNMAN RULES FOR QED

- draw the topologically distinct connected diagrams.
- vertex factor = $e \gamma^\mu$; 4-momentum is conserved at the vertex.
- fermion propagator = $S_F(p)$
- photon propagator = $D_{\mu\nu}(q)$
- external fermion leg = a Dirac spinor; $u(p, s), \bar{u}(p, s), v(p, s), \bar{v}(p, s)$
- external photon leg = a polarization 4-vector; $\epsilon_\mu(q)$
- verify the relative signs!

$\text{E E-bar} \rightarrow \text{Mu Mu-bar}$

$e^-(p_1) + e^+(p_2) \rightarrow \mu^-(p_3) + \mu^+(p_4)$

$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$

There is only one Feynman diagram

Thus the transition matrix element is

$\mathcal{M} = \overline{u}(p_2) e^\gamma_\mu u(p_1) D_{\mu\nu}(p_1 + p_2)$
The goal is to calculate the cross section. Recall the steps in the calculation.

☆ Square $M$

$$\begin{align*}
|M|^2 &= \frac{e^4}{s^2} \bar{u}_2 \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_2 \\
&\quad \cdot \frac{s_3}{s_4} \gamma_\mu \gamma_\nu u_3 \\
&\quad \cdot \frac{1}{4} \left( \bar{u}_2 \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_2 \\
&\quad + \bar{u}_2 \gamma^\nu u_1 \bar{u}_1 \gamma^\mu u_2 \right)
\end{align*}$$

because

$$\gamma^0 \gamma^+ \gamma^0 = \gamma^0 \quad \text{for} \quad \nu = 0$$

$$\gamma^0 \gamma^- \gamma^0 = \gamma^0 \quad \text{for} \quad \nu = 0$$

$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu$$

☆ For unpolarized annihilation, average over $s_1$ and $s_2$, and sum over $s_3$ and $s_4$.

$$\begin{align*}
\overline{|M|^2} &= \frac{e^4}{s^2} \frac{1}{4} \left( \overline{\bar{u}_2 \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_2} \\
&\quad + \bar{u}_2 \gamma^\nu u_1 \bar{u}_1 \gamma^\mu u_2 \right) \\
&\quad \cdot \left( \bar{u}_2 \gamma_\mu \gamma_\nu u_1 \bar{u}_1 \gamma_\mu \gamma_\nu u_2 \\
&\quad + \bar{u}_2 \gamma_\nu \gamma_\mu u_1 \bar{u}_1 \gamma_\nu \gamma_\mu u_2 \right)
\end{align*}$$

☆ Calculate the traces
Traces

\[ E^{\mu \nu} = \text{Tr} \, g^{\mu} (\gamma_{\mu} + m) g^{\nu} (\gamma_{\nu} - m) \]

\[ = \text{Tr} \, g^{\mu} \gamma_{\mu} \gamma_{\nu} \gamma_{\nu} - m^2 \text{Tr} \, g^{\mu} \gamma_{\mu} \gamma_{\nu} \gamma_{\nu} \]

\[ = 4 \, p_{1}^{\mu} p_{1}^{\nu} - 4 \, g^{\mu \nu} p_{1} \cdot p_{2} + 4 \, g^{\mu \nu} \]

\[ - m^2 \, 4 \, g^{\mu \nu} \]

\[ = 4 \left( p_{1}^{\mu} p_{2}^{\nu} + p_{2}^{\mu} p_{1}^{\nu} \right) \]

\[ - 4 \, g^{\mu \nu} (p_{1} \cdot p_{2} + m^2) \]

\[ M_{\mu \nu} = \text{Tr} \, g^{\mu} (\gamma_{\mu} + M) g^{\nu} (\gamma_{\nu} + M) \]

\[ = 4 \left( p_{3}^{\mu} p_{3}^{\nu} + p_{4}^{\mu} p_{4}^{\nu} \right) \]

\[ - 4 \, g^{\mu \nu} (p_{3} \cdot p_{4} + M^2) \]

\[ E^{\mu \nu} M_{\mu \nu} \]

\[ = 16 \left( p_{1} \cdot p_{4} p_{2} \cdot p_{3} \times 2 + p_{1} \cdot p_{3} p_{2} \cdot p_{4} \times 2 \right) \]

\[ - 16 \left( p_{1} \cdot p_{2} \times 2 \left( p_{3} \cdot p_{4} + M^2 \right) \right) \]

\[ - 16 \left( p_{3} \cdot p_{4} \times 2 \left( p_{1} \cdot p_{2} + m^2 \right) \right) \]

\[ + 16 \cdot 4 \left( p_{1} \cdot p_{2} + m^2 \right) \left( p_{3} \cdot p_{4} + M^2 \right) \]

To simplify, substitute the Mandelstam variables —

\[ S = (p_{1} + p_{2})^2 = 2M^2 + 2p_{1} \cdot p_{2} \]

\[ S = (p_{3} + p_{4})^2 = 2M^2 + 2p_{3} \cdot p_{4} \]

\[ t = (p_{1} - p_{3})^2 = m^2 + M^2 - 2p_{1} \cdot p_{3} \]

\[ t = (p_{4} - p_{2})^2 = m^2 + M^2 - 2p_{2} \cdot p_{4} \]

\[ u = (p_{1} - p_{4})^2 = m^2 + M^2 - 2p_{1} \cdot p_{4} \]

\[ u = (p_{3} - p_{2})^2 = m^2 + M^2 - 2p_{3} \cdot p_{2} \]
Substitute Mandelstam variables (See page 156)

\[ E^{\mu\nu} M_{\mu\nu} = 8 \left( m^2 + M^2 - u \right)^2 + 8 \left( m^2 + M^2 - t \right)^2 - 32 p_1 \cdot p_2 \frac{8}{s_2} - 32 \frac{1}{s_2} \cdot \frac{8}{s_2} + 64 \frac{s_1}{s_2} \cdot \frac{s_2}{s_2} + 32 \left( m^2 + M^2 \right) \frac{s_1}{s_2} \]

Result

\[
\frac{1}{M^2} = \frac{e^4}{4s^2} E^* M
\]

\[
= \frac{e^4}{4s^2} \left\{ 8 \left( m^2 + M^2 - u \right)^2 + \left( m^2 + M^2 - t \right)^2 + 2 \left( m^2 + M^2 \right) \frac{s_1}{s_2} \right\}
\]

\[
= \frac{2e^4}{s^2} \left\{ 8 \left( m^2 + M^2 - u \right)^2 + \left( m^2 + M^2 - t \right)^2 + 2 \left( m^2 + M^2 \right) \frac{s_1}{s_2} \right\}
\]

Note: This is Lorentz invariant and dimensionless.

The center-of-mass differential cross section
The cross section

\[ \sigma = \frac{1}{4 \pi^2 \gamma^2} \frac{d^3 p_3}{d^3 p_1} \int \frac{d^3 p_4}{d^3 p_2} \left( \frac{d^3 p_5}{d^3 p_6} \right) \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{q}) \delta^3(\mathbf{p}_4 + \mathbf{p}_5 - \mathbf{q}_2) \]

\[ \delta(\mathbf{E}_3 - \mathbf{E}_1 - \mathbf{E}_2) \delta(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{q}_1) \]

In terms of the energy \( E_1 \)

the energy of one of the initial electrons,

\[ \frac{d\sigma}{dE_2} = \frac{\frac{1}{2}M^2}{(2\pi)^2 \left( \frac{E_1}{2} \right)^4} \frac{1}{4E_1} \delta^2 \left( \frac{E_2}{E_1} - 1 \right)^2 \]

Now write \( \frac{1}{2}M^2 \) in terms of \( E_1 \) and \( \theta \):

\[ \frac{1}{2}M^2 = \frac{2e^4}{s^2} \left\{ \left( \frac{w^2 + \mathbf{H}^2 - \mathbf{u}^2}{2} \right)^2 + \left( \frac{w^2 + \mathbf{H}^2 - \mathbf{t}^2}{2} \right)^2 + 2 \left( \frac{w^2 + \mathbf{H}^2}{2} \right)^2 \right\} \]

\[ = \frac{2e^4}{s^2} \left\{ \left( \frac{2E_1^2 + 2w_1 \mathbf{p}_3 \cos \theta}{2} \right)^2 + \left( \frac{2E_1^2 - 2w_1 \mathbf{v}_2 \cos \theta}{2} \right)^2 + 2 \left( \frac{w^2 + \mathbf{H}^2}{2} \right)^2 \right\} \]
Further simplifications

We can approximate $m = 0$ because $m << M$ and $m << E$. Then the result is PS equation 5.12.
Electron - positron annihilation to hadrons
These were important experiments in the history of high-energy physics.
From the particle data group, the figure shows the ratio
\[ R = \frac{\sigma (e^- e^+ \rightarrow \text{hadrons})}{\sigma (e^- e^+ \rightarrow \mu^- \mu^+)} \, . \]
The underlying process in hadron production is \( e^- + e^+ \rightarrow q + q \bar{q} \).
Neglecting QCD interactions we would just have \( R = \text{constant} \) (green = naive quark model; red = QCD 3rd order).
The $\sqrt{s}$ dependence is due to 2 effects:

- thresholds
- resonances

But between thresholds we do indeed have $R \approx$ constant.

And we can even estimate the constant:

above the b-quark threshold,

$$R = \sum q e^2 / e^2 = \left( \frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} \right) \times 3 = \frac{11}{3} = 3.67.$$  

This is a great success of QCD.

Historically, measurements of $R$ provided evidence for the quark model and QCD.