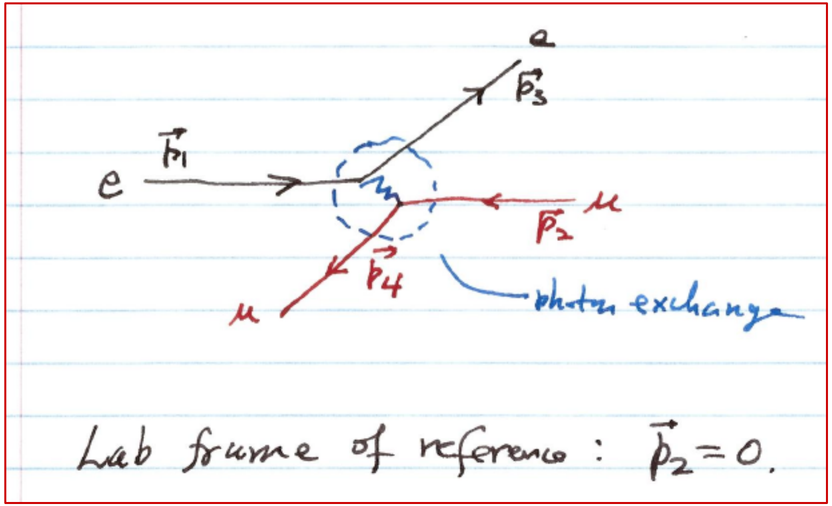
Electron-muon scattering

Imagine the process

$$e(p_1) + \mu(p_2) \rightarrow e(p_3) + \mu(p_4)$$

$$p_1^{\mu} + p_2^{\mu} = p_3^{\mu} + p_4^{\mu}$$

We will regard the electron as the projectile and the muon as the target.



Comments.

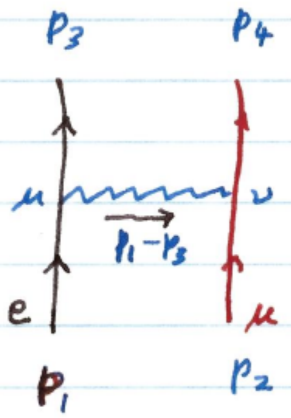
This is an imaginary process, because the muon is an unstable particle (mean lifetime in the muon rest frame = 2.2 μs) so you can't make a muon target. We'll do the calculation, imagining that the muon is stable, as an academic exercise -- a simple example in QED.

One could make eμ scattering more physical by regarding the muon as the projectile and the electron as the target.

E.g., consider a muon created at the top of the atmosphere by a cosmic ray. How does it lose energy (by ionization of molecules) as it passes through the atmosphere. Or, how does a muon make a track (by ionization) in a muon detector?

(1.) The transition amplitude

There is only one Feynman diagram.



$$\mathcal{M} = \bar{u}(p_3)\, e\gamma^\mu\, u(p_1)$$
$$\bar{u}(p_4)\, e\gamma^\nu\, u(p_2)$$
$$D_{\mu\nu}(p_1 - p_3)$$

$$D_{\mu\nu}(q) = \frac{g_{\mu\nu}}{q^2}$$

$$\mathcal{M} = \frac{e^2}{(p_1 - p_3)^2}\, \bar{u}_3 \gamma^\mu u_1 \;\; \bar{u}_4 \gamma_\mu u_2$$

(2.) Calculate the magnitude squared

$$|\mathcal{M}|^2 = \left(\frac{e^2}{t}\right)^2 \; \bar{u}_3 \gamma^\mu u_1 \;\; \bar{u}_1 \gamma^\nu u_3$$
$$\bar{u}_4 \gamma_\mu u_2 \;\; \bar{u}_2 \gamma_\nu u_4$$

where $t = (p_1 - p_3)^2$

$$|\mathcal{M}|^2 = \frac{e^4}{t^2} \; \mathrm{Tr}\; \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_3 \bar{u}_3$$
$$\mathrm{Tr}\; \gamma_\mu u_2 \bar{u}_2 \gamma_\nu u_4 \bar{u}_4$$

(3.) For unpolarized scattering...

... we average over the initial spins and sum over the final spins.

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{t^2}\,\frac{1}{4}\,\mathrm{Tr}\,\gamma^\mu(\not p_1 + m)\gamma^\nu(\not p_3 + m)$$

$$\mathrm{Tr}\,\gamma_\mu(\not p_2 + M)\gamma_\nu(\not p_4 + M)$$

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4t^2} E^{\mu\nu} M_{\mu\nu}$$

(4.) Calculate the traces.

$$E^{\mu\nu} = \mathrm{Tr}\,\gamma^\mu(\not p_1 + m)\gamma^\nu(\not p_3 + m)$$

$$= 4p_1^\mu p_3^\nu - 4g^{\mu\nu} p_1\cdot p_3 + 4p_3^\mu p_1^\nu$$

$$+ m^2 4g^{\mu\nu}$$

$$= 4\left[p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2 - p_1\cdot p_3)\right]$$

$$M_{\mu\nu} = 4\left[p_{2\mu}p_{4\mu} + p_{4\mu}p_{2\nu} + g_{\mu\nu}(M^2 - p_2\cdot p_4)\right]$$

$$E\cdot M = E^{\mu\nu}M_{\mu\nu}$$

$$= 16\left[2\,p_1\cdot p_2\,p_3\cdot p_4 + 2\,p_1\cdot p_4\,p_2\cdot p_3\right]$$

$$+16\left[2p_1\cdot p_3\,(M^2 - p_2\cdot p_4)\right.$$

$$\left.+(m^2 - p_1\cdot p_3)\,2p_2\cdot p_4\right]$$

$$+16\cdot 4\cdot(m^2 - p_1\cdot p_3)(M^2 - p_2\cdot p_4)$$

(5.) Substitute Mandelstam variables

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$

$$\therefore \quad p_1\cdot p_2 = p_3\cdot p_4 = (s-m^2-M^2)/2$$

$$t = (p_1-p_3)^2 = (p_4-p_2)^2$$

$$\therefore \quad p_1\cdot p_3 = (2m^2-t)/2$$

$$\therefore \quad p_2\cdot p_4 = (2M^2-t)/2$$

$$u = (p_1-p_4)^2 = (p_3-p_2)^2$$

$$\therefore \quad p_1\cdot p_4 = p_2\cdot p_3 = (m^2+M^2-u)/2$$

step 1 : E dot M in Mandelstam variables

```
In[306]:= (* e mu → e mu *)
EdM = 16 * (2 * p1dp2 * p3dp4 + 2 * p1dp4 * p2dp3) +
    (-1) * 16 * (p1dp3 - m^2) * 2 * p2dp4 +
    (-1) * 16 * (p2dp4 - M^2) * 2 * p1dp3 +
    16 * 4 * (p1dp3 - m^2) * (p2dp4 - M^2);
all = {p1dp2 -> (s - m^2 - M^2) / 2, p3dp4 -> (s - m^2 - M^2) / 2,
    p1dp3 -> (2 * m^2 - t) / 2, p2dp4 -> (2 * M^2 - t) / 2,
    p1dp4 -> (m^2 + M^2 - u) / 2, p2dp3 -> (m^2 + M^2 - u) / 2};

In[314]:= EdM = EdM /. all ;
EdM = FullSimplify[EdM]
Expand[EdM /. {s → m^2 + M^2 + s0, u → m^2 + M^2 + u0}]
```

Out[315]= $8\left(2m^4+2M^4+s^2+2m^2\left(2M^2-s+t-u\right)+u^2-2M^2\left(s-t+u\right)\right)$

Out[316]= $8\,s0^2+16\,m^2\,t+16\,M^2\,t+8\,u0^2$

$$E\cdot M = 8\Big[(s-m^2-M^2)^2 + (m^2+M^2-u)^2 + 2(m^2+M^2)\,t\Big]$$

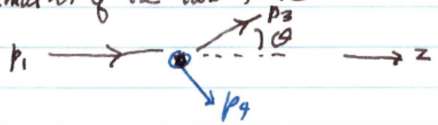
Note that $|M|^2$ is dimensionless and Lorentz invariant.

(6.) Calculate the differential cross section...

...in the lab frame of reference;
i.e. the *rest frame* of the initial muon

$$d\sigma = \frac{1}{2E_1 \cdot 2E_2 v_{rel}} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} \overline{|\mathcal{M}|^2}$$
$$(2\pi)^4 \, \delta(E_3+E_4-E_1-E_2)\, \delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)$$

Kinematics of the lab frame



$$p_1^\mu = (E_1,\ 0,\ 0,\ p_1)$$
$$p_2^\mu = (M,\ 0,\ 0,\ 0)$$
$$p_3^\mu = (E_3,\ p_3\sin\theta,\ 0,\ p_3\cos\theta)$$
$$p_4^\mu = (E_4,\ -p_3\sin\theta,\ 0,\ p_1 - p_3\cos\theta)$$

Energy conservation:

$$\sqrt{p_3^2+m^2} + \sqrt{|\vec{p}_1-\vec{p}_3|^2+M^2} = \sqrt{p_1^2+m^2} + M$$

Approximation: $m_e = 0$

accurate for energies E1 >> 1 MeV

Then the equation for energy conservation becomes

$$p_3 + \sqrt{(\vec{p}_1-\vec{p}_3)^2+M^2} = p_1 + M$$
$$(\vec{p}_1-\vec{p}_3)^2 + M^2 = (p_1+M-p_3)^2$$
$$p_1^2+p_3^2-2p_1p_3\cos\theta+M^2 = p_1^2+M^2+p_3^2 + 2M(p_1-p_3) - 2p_1p_3$$
$$2p_1p_3(1-\cos\theta) = 2M(p_1-p_3)$$
$$p_3\left[M+p_1(1-\cos\theta)\right] = Mp_1$$
$$p_3 = \frac{Mp_1}{M+p_1(1-\cos\theta)} = \frac{Mp_1}{\Delta}$$

step 2 : lab frame kinematics

```
In[320]:= (* lab frame (initial muon at rest) *)
Ms = {s → m^2 + M^2 + 2 * E1 * M,
   t → -2 * M * (E1 - E3),
   u → m^2 + M^2 - 2 * M * E3};
EdM = EdM /. Ms;
EdM = FullSimplify[EdM]
```

Out[322]= $32\,M\left(E1^2\,M - E1\left(m^2+M^2\right) + E3\left(m^2 + M\left(E3+M\right)\right)\right)$

step 3 : electron mass = 0

```
In[323]:= (* approximate m = 0 *)
appx = {m → 0, E1 → p1, E3 → p3};
EdM = EdM /. appx;
EdM = FullSimplify[EdM]
EdM /. {M → 106., p1 → 10, p3 → 106 * 10 / (106 + 10)}
```

Out[325]= $32\,M^2\left(-M\,p1 + p1^2 + p3\left(M+p3\right)\right)$

Out[326]= 3.31228×10^{7}

step 4 : $|ME|^2$

Substitute p3 -> M p1 /Δ
{where I define Δ = M + p1 * (1 - cos θ)}

```
In[327]:= EdM = EdM /. {p3 → M * p1 / Δ};
EdM = FullSimplify[EdM]
```

Out[328]= $32\,M^2\,p1\left(-M + p1 + \frac{M^2\left(p1+\Delta\right)}{\Delta^2}\right)$

```
In[329]:= Msq = EdM * e^4 / (4 * t^2);
Msq = Msq /. {t → -2 * M * p1 (1 - M / Δ)};
Msq = FullSimplify[Msq]
```

Out[331]= $\frac{2\,e^4\left(-M\,\Delta^2 + p1\,\Delta^2 + M^2\left(p1+\Delta\right)\right)}{p1\left(M-\Delta\right)^2}$

$$d\sigma = \frac{1}{2E_1 \cdot 2E_2 v_{rel}} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4} \overline{|\mathcal{M}|^2}$$

$$(2\pi)^4 \, \delta(E_3+E_4-E_1-E_2)\, \delta^3(\vec{p}_3+\vec{p}_4-\vec{p}_1-\vec{p}_2)$$

In the lab frame :

$E_2 = M$; $v_{rel} = \frac{p_1}{E_1}$; $E_4 = E_1 + M - E_3$

With the approximation $m=0$:

$v_{rel} = 1$; $E_1 = p_1$, $E_3 = p_3$;

$E_4 = p_1 - p_3 + M$

$$\sigma = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^2} \frac{1}{4Mp_1} \int \frac{p_3^2\, dp_3\, d\Omega_3}{\cancel{4p_3}\, E_4}$$

$$\int dp_3\, \delta\left(p_3 + \sqrt{(\vec{p}_1-\vec{p}_3)^2+M^2} - E_1 - M \right)$$

$$\int dp_3\, \delta[f(p_3)] = \frac{1}{|f'(p_3)|}$$

$$f(p_3) = p_3 + \sqrt{p_1^2 + p_3^2 - 2p_1 p_3 \cos\theta + M^2} - E_1 - M$$

$$f'(p_3) = 1 + \frac{1}{2}\frac{1}{E_4}(2p_3 - 2p_1\cos\theta)$$

$$= \frac{1}{E_4}\{E_4 + p_3 - p_1\cos\theta\}$$

$$= \frac{1}{E_4}\{M + p_1(1-\cos\theta)\}$$

$$= \Delta / E_4$$

$$\frac{d\sigma}{d\Omega_3} = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^2} \frac{1}{4Mp_1} \frac{p_3^2}{4p_3 E_4} \frac{E_4}{\Delta}$$

$$\frac{d\sigma}{d\Omega_3} = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^2} \frac{p_3}{16Mp_1\Delta}$$

where again $\Delta = M + p_1(1-\cos\theta)$
and $p_3 = Mp_1/\Delta$

$$\boxed{\frac{d\sigma}{d\Omega_3} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2\Delta^2}}$$

step 5 : cross section

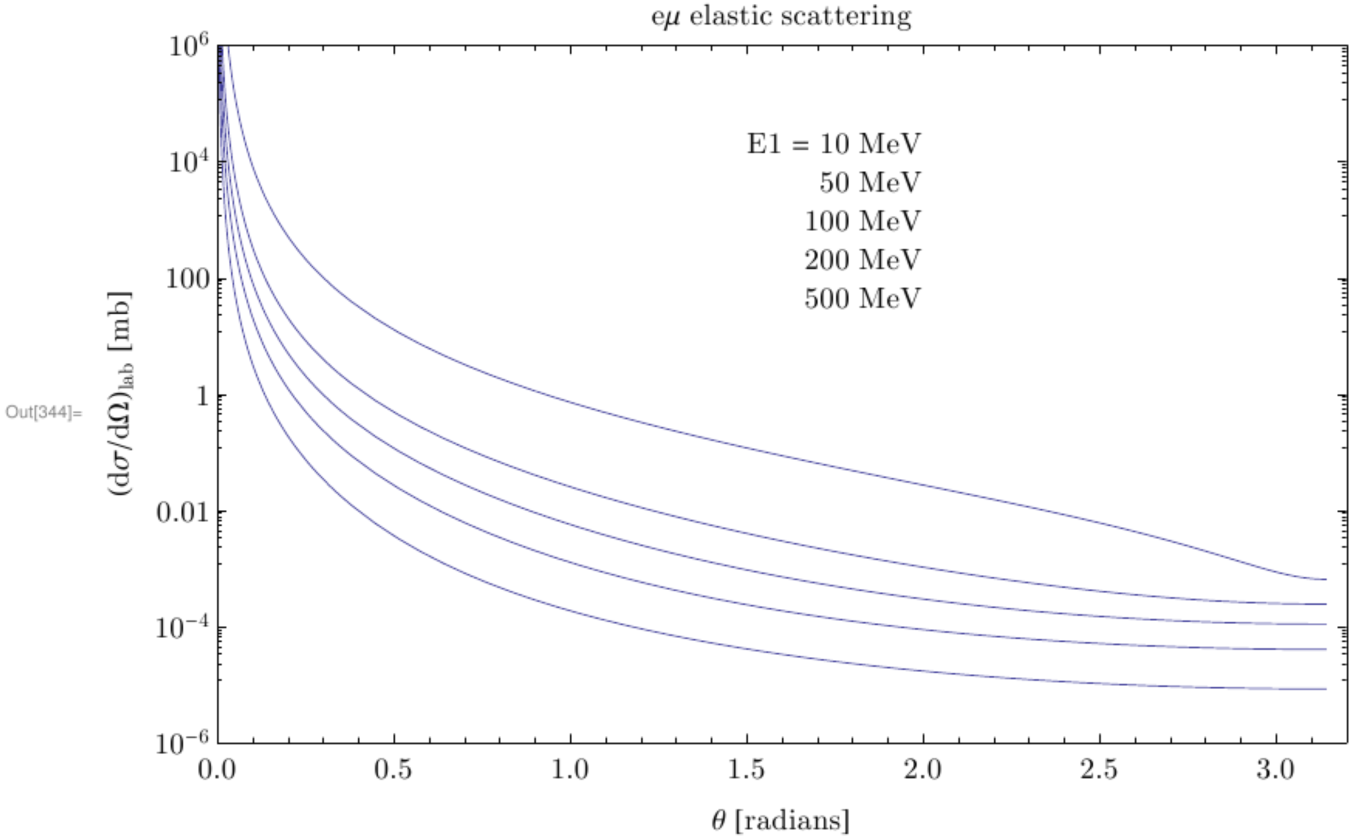
```
In[332]:= cs = Msq / (64 * Pi ^ 2) / Δ ^ 2;
          cs = FullSimplify[cs];
          cs = cs /. {Δ → M + p1 * (1 - costh)};
          cs = FullSimplify[Expand[cs]]
```

Out[335]=
$$\frac{e^4\,\left((1+\mathrm{costh})\,M^2+(-1+\mathrm{costh})^2\,\mathrm{p1}^2+M\,(\mathrm{p1}-\mathrm{costh}^2\,\mathrm{p1})\right)}{32\,(-1+\mathrm{costh})^2\,\mathrm{p1}^2\,(M+\mathrm{p1}-\mathrm{costh}\,\mathrm{p1})^2\,\pi^2}$$

```
In[336]:= cross = cs /. {e → 1};
          cross = cross * (4 * Pi / 137) ^ 2 * 197 ^ 2 * ( 1000 / 100) ;(* 1 b = 100 fm^2 *)
          σ[E1_, th_, Mmu_] := cross /. {p1 → E1, costh → Cos[th], M → Mmu}
          σ[E1, θ, Mmu]
          {σ[10, 3.14 / 2, 106], σ[200, 3.14 / 2, 106]}
```

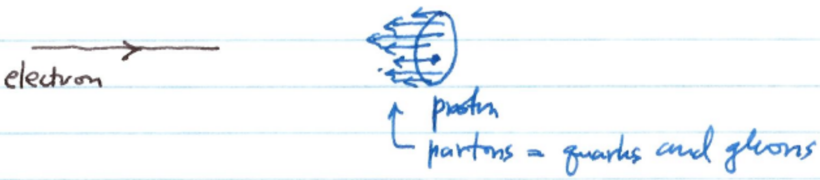
Out[339]=
$$\frac{194\,045\,\left(\mathrm{E1}^2\,(-1+\mathrm{Cos}[\theta])^2+\mathrm{Mmu}^2\,(1+\mathrm{Cos}[\theta])+\mathrm{Mmu}\,(\mathrm{E1}-\mathrm{E1}\,\mathrm{Cos}[\theta]^2)\right)}{18\,769\,\mathrm{E1}^2\,(-1+\mathrm{Cos}[\theta])^2\,(\mathrm{E1}+\mathrm{Mmu}-\mathrm{E1}\,\mathrm{Cos}[\theta])^2}$$

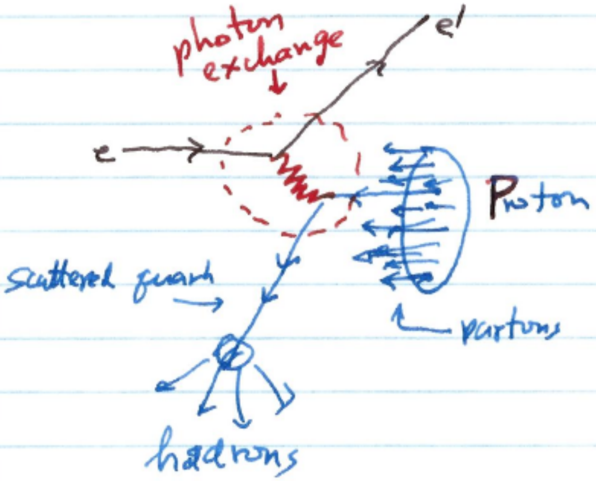
Out[340]= {0.0954743, 0.000200322}



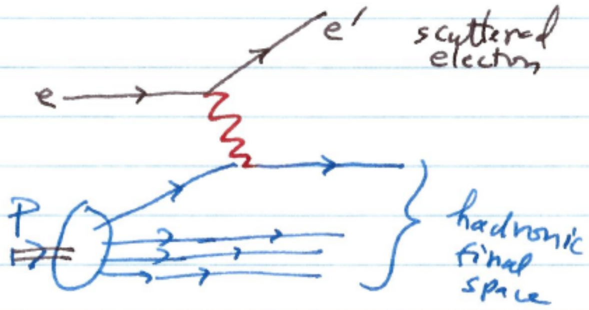
Out[344]=
eμ elastic scattering
E1 = 10 MeV
50 MeV
100 MeV
200 MeV
500 MeV
10^6
10^4
100
1
0.01
10^{-4}
10^{-6}
$(d\sigma/d\Omega)_{lab}$ [mb]
0.0
0.5
1.0
1.5
2.0
2.5
3.0
θ [radians]

Electron Proton deep inelastic scattering





Parton Distribution Functions



$$d\sigma_p = \int_0^1 f_q(x)\; d\sigma_q(x)\; dx$$

x = momentum fraction

MEASURE

EXTRACT from DATA

CALCULATE