Chapter 6 - Radiative Corrections

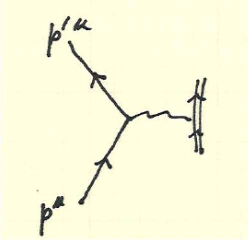
There are two kinds of radiative corrections.

Virtual corrections:
Feynman diagrams have loops.

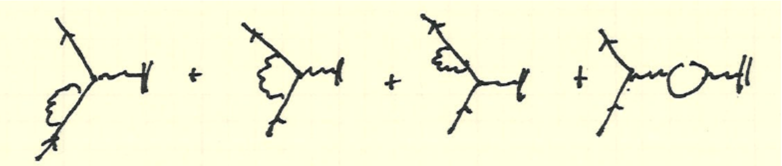
Real corrections:
Feynman diagrams have additional final state particles, i.e, photons.

We will see that ***these two kinds of corrections cannot be separated.***
At first sight this seems very surprising. But on further thought, it's not so surprising.

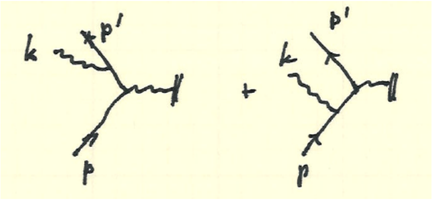
Example. The lowest order corrections to electron scattering.
Tree diagram (LO of perturbation theory)



Virtual corrections (NLO → loops)



Real radiation



We'll see that the real and virtual effects must be combined.

Divergences

There are two kinds of divergences (actually three[†] but we'll only consider two)

UV divergences

The integral over an internal loop momentum qμ may be divergent (i.e., infinite) due to the contribution from large q^μ. For example,

$$\int \frac{d^4 q}{(q^2+m^2)^2} \sim \int \frac{dq}{q}$$

"Miraculously", the UV divergences cancel out after RENORMALIZATION.

[† the third kind are *collinear divergences*]

IR divergences

(i) The integral over an internal loop momentum qμ may be divergent (i.e., infinite) due to the contribution from small q^μ. For example,

$$\int \frac{d^4 q}{q^2 (q^2+m^2)^2}$$

(ii) Also, the integral over final states in a real radiative process, e.g., Bremsstrahlung, may be divergent due to the contribution from small q^μ.

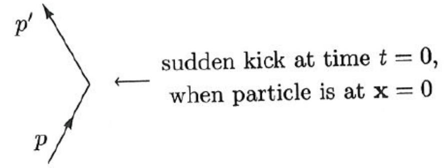
"Miraculously", the IR divergences cancel out after COMBINING REAL AND VIRTUAL CORRECTIONS.

SECTION 6.1. Bremsstrahlung of soft photons.

What is a *soft* photon?
A soft photon is a low-energy photon. now, how low is low? that depends on the process. a photon that is emitted in a scattering process with characteristic momentum/energy scale Q is considered to be a soft photon if ω << Q.

CLASSICAL CALCULATION

For details, see P.&Sch.



The electron trajectory is

$$\boxed{j^\mu(x) = e\int d\tau\, \frac{dy^\mu(\tau)}{d\tau}\, \delta^{(4)}(x - y(\tau)).}$$

$$\boxed{y^\mu(\tau) = \begin{cases} (p^\mu/m)\tau & \text{for } \tau < 0; \\ (p'^\mu/m)\tau & \text{for } \tau > 0. \end{cases}}$$

Use ***classical*** electrodynamics to calculate the vector potential

$$\boxed{\widetilde{A}^\mu(k) = -\frac{1}{k^2}\,\tilde{j}^\mu(k).}$$

$$\boxed{A^\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \frac{-ie}{k^2}\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right).}$$

And use the vector potential to calculate the radiation fields (**E** and **B**).

$$\mathcal{E}(\mathbf{k}) = -i\mathbf{k}\mathcal{A}^0(\mathbf{k}) + ik^0\boldsymbol{\mathcal{A}}(\mathbf{k});$$

$$\boldsymbol{\mathcal{B}}(\mathbf{k}) = i\mathbf{k} \times \boldsymbol{\mathcal{A}}(\mathbf{k}) = \hat{k} \times \boldsymbol{\mathcal{E}}(\mathbf{k}).$$

$$\text{Energy} = \tfrac{1}{2}\int d^3x\,\left(|\mathbf{E}(x)|^2 + |\mathbf{B}(x)|^2\right).$$

Using the explicit form of $\boldsymbol{\mathcal{A}}(\mathbf{k})$ (6.7), we finally arrive at an expression for the energy radiated*:

$$\text{Energy} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \frac{e^2}{2}\left|\epsilon_\lambda(\mathbf{k}) \cdot \left(\frac{\mathbf{p}'}{k\cdot p'} - \frac{\mathbf{p}}{k\cdot p}\right)\right|^2. \qquad (6.12)$$

$$\text{Energy} = \frac{e^2}{(2\pi)^2} \int dk \, \mathcal{I}(\mathbf{v}, \mathbf{v}'),$$

$$\mathcal{I}(\mathbf{v}, \mathbf{v}') = \int \frac{d\Omega_{\hat{k}}}{4\pi} \left(\frac{2(1-\mathbf{v}\cdot\mathbf{v}')}{(1-\hat{k}\cdot\mathbf{v})(1-\hat{k}\cdot\mathbf{v}')} - \frac{m^2/E^2}{(1-\hat{k}\cdot\mathbf{v}')^2} - \frac{m^2/E^2}{(1-\hat{k}\cdot\mathbf{v})^2} \right).$$

$$\mathcal{I}(\mathbf{v}, \mathbf{v}') \approx \log\left(\frac{1-\mathbf{v}'\cdot\mathbf{v}}{1-|\mathbf{v}|}\right) + \log\left(\frac{1-\mathbf{v}'\cdot\mathbf{v}}{1-|\mathbf{v}'|}\right) = \log\left(\frac{(E^2-\mathbf{p}\cdot\mathbf{p}')^2}{E^2(E-|\mathbf{p}|)^2}\right)$$

$$\approx 2\log\left(\frac{p\cdot p'}{(E^2-|\mathbf{p}|^2)/2}\right) = 2\log\left(\frac{-q^2}{m^2}\right),$$

where $q^2 = (p'-p)^2$.

Result of the classical calculation

In conclusion, we have found that the radiated energy at low frequencies is given by

$$\text{Energy} = \frac{\alpha}{\pi}\int_0^{k_{\max}} dk\, \mathcal{I}(\mathbf{v},\mathbf{v}') \underset{E\gg m}{\longrightarrow} \frac{2\alpha}{\pi}\int_0^{k_{\max}} dk\, \log\left(\frac{-q^2}{m^2}\right). \tag{6.18}$$

If this energy is made up of photons, each photon contributes energy k. We would then expect

$$\text{Number of photons} = \frac{\alpha}{\pi}\int_0^{k_{\max}} dk\, \frac{1}{k}\mathcal{I}(\mathbf{v},\mathbf{v}'). \qquad \text{SC} \tag{6.19}$$

We hope that a quantum-mechanical calculation will confirm this result.

Of course there are no photons in classical electromagnetism.
We need a QCD calculation to find the number of photons.

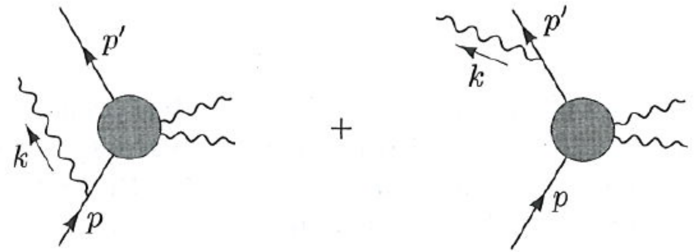
Q.E.D. CALCULATION

For single photon emission, there are two Feynman diagrams in lowest order.

Now we'll make lots of approximations, which are valid for soft photons; i.e. the asymptotic behavior as $\omega \to 0$. (Not strictly 0! But small compared to the momentum transfer that occurs in the scattering process.)

Quantum Computation

Consider now the quantum-mechanical process in which one photon is radiated during the scattering of an electron:



Let $\mathcal{M}_0$ denote the part of the amplitude that comes from the electron's interaction with the external field. Then the amplitude for the whole process is

$$i\mathcal{M} = -ie\bar{u}(p')\bigg(\mathcal{M}_0(p', p-k)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\gamma^\mu\epsilon^*_\mu(k) + \gamma^\mu\epsilon^*_\mu(k)\frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2}\mathcal{M}_0(p'+k, p)\bigg)u(p). \quad (6.20)$$

$$\mathcal{M}_0(p', p-k) \approx \mathcal{M}_0(p'+k, p) \approx \mathcal{M}_0(p', p),$$

and we can ignore $\not{k}$ in the numerators of the propagators. The numerators can be further simplified with some Dirac algebra. In the first term we have

$$\begin{aligned}(\not{p}+m)\gamma^\mu \epsilon^*_\mu u(p) &= \left[2p^\mu \epsilon^*_\mu + \gamma^\mu \epsilon^*_\mu(-\not{p}+m)\right]u(p) \\ &= 2p^\mu \epsilon^*_\mu u(p).\end{aligned}$$

Similarly, in the second term,

$$\bar{u}(p')\,\gamma^\mu \epsilon^*_\mu(\not{p}'+m) = \bar{u}(p')\,2p'^\mu \epsilon^*_\mu.$$

The denominators of the propagators also simplify:

$$(p-k)^2 - m^2 = -2p\cdot k; \qquad (p'+k)^2 - m^2 = 2p'\cdot k.$$

So in the soft-photon approximation, the amplitude becomes

$$i\mathcal{M} = \bar{u}(p')\left[\mathcal{M}_0(p',p)\right]u(p)\cdot\left[e\left(\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right)\right]. \qquad (6.22)$$

This is just the amplitude for elastic scattering (without bremsstrahlung), times a factor (in brackets) for the emission of the photon.

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \cdot \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \sum_{\lambda=1,2} e^2 \left| \frac{p' \cdot \epsilon^{(\lambda)}}{p' \cdot k} - \frac{p \cdot \epsilon^{(\lambda)}}{p \cdot k} \right|^2 . \quad (6.23)$$

$$d(\text{prob}) = \frac{d^3k}{(2\pi)^3} \sum_\lambda \frac{e^2}{2k} \left| \boldsymbol{\epsilon}_\lambda \cdot \left(\frac{\mathbf{p}'}{p' \cdot k} - \frac{\mathbf{p}}{p \cdot k} \right) \right|^2 .$$

$$\text{Total probability} \approx \frac{\alpha}{\pi} \int_0^{|\mathbf{q}|} dk \, \frac{1}{k} \, \mathcal{I}(\mathbf{v}, \mathbf{v}').$$

An infrared divergence

We can artificially make the integral in (6.25) well-defined by pretending that the photon has a very small mass μ. This mass would then provide a lower cutoff for the integral, allowing us to write the result of this section as

$$\begin{aligned} d\sigma(p \to p' + \gamma(k)) &= d\sigma(p \to p') \cdot \frac{\alpha}{2\pi} \log\left(\frac{-q^2}{\mu^2}\right) \mathcal{I}(\mathbf{v}, \mathbf{v}') \\ &\underset{-q^2 \to \infty}{\approx} d\sigma(p \to p') \cdot \frac{\alpha}{\pi} \log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right). \end{aligned} \quad (6.26)$$

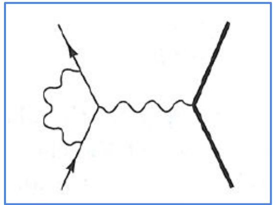
Result of the quantum calculation

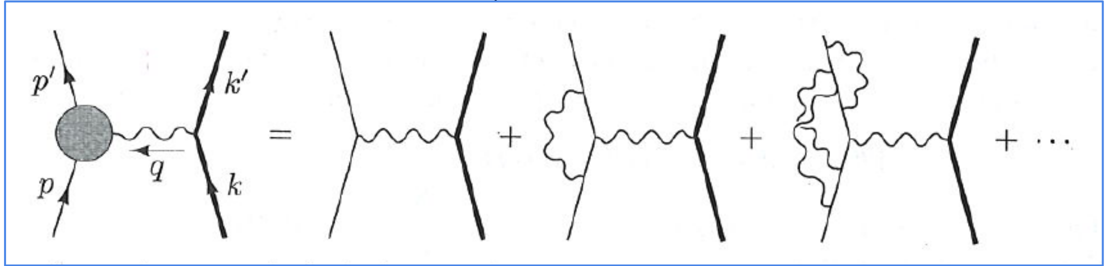
Does this agree with the semiclassical result (SC) derived earlier?

The two results are sort of similar mathematically; but they are actually very different physically.

In Section 6.5 we'll see that the semiclassical result is correct, when we add all the higher order contributions to soft photon bremsstrahlung.

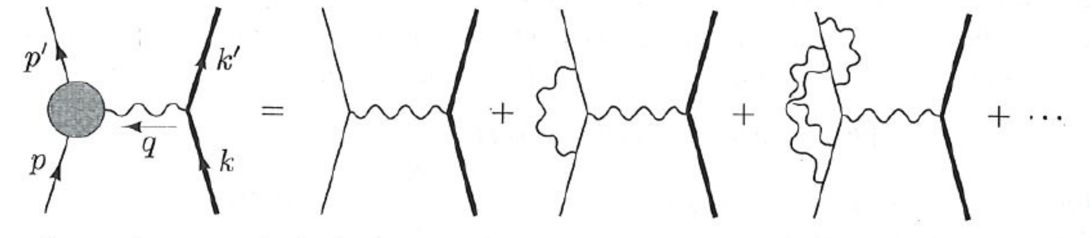
Section 6.2. The electron vertex function





$$i\mathcal{M} = ie^2\left(\bar{u}(p')\,\Gamma^\mu(p',p)\,u(p)\right)\frac{1}{q^2}\left(\bar{u}(k')\gamma_\mu u(k)\right).$$

Eventually we'll calculate the lowest order vertex correction. But first let's understand some general features of the correction, valid to all orders of perturbation theory.



$$i\mathcal{M}\,(2\pi)\delta(p^{0\prime} - p^0) = -ie\bar{u}(p')\gamma^\mu u(p) \cdot \tilde{A}^{\text{cl}}_\mu(p' - p),$$

$$i\mathcal{M}\,(2\pi)\delta(p^{0\prime} - p^0) = -ie\bar{u}(p')\,\Gamma^\mu(p',p)\,u(p) \cdot \tilde{A}^{\text{cl}}_\mu(p' - p).$$

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2),$$

Electric and Magnetic Form Factors

$\boxed{\text{limit } \mathbf{q} \to 0}$ in the spinor matrix element. Only the form factor F_1 contributes. Using the nonrelativistic limit of the spinors,

$$\bar{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) \approx 2m\xi'^\dagger \xi,$$

the amplitude for electron scattering from an electric field takes the form

$$\boxed{i\mathcal{M} = -ieF_1(0)\tilde{\phi}(\mathbf{q}) \cdot 2m\xi'^\dagger \xi.} \tag{6.34}$$

This is the Born approximation for scattering from a potential

$$\boxed{V(\mathbf{x}) = eF_1(0)\phi(\mathbf{x}).}$$

Thus $F_1(0)$ is the electric charge of the electron, in units of e. Since $F_1(0) = 1$ already in the leading order of perturbation theory, radiative corrections to $F_1(q^2)$ should vanish at $q^2 = 0$.

Again we can interpret $\mathcal{M}$ as the Born approximation to the scattering of the electron from a potential well. The potential is just that of a magnetic moment interaction,

$$V(\mathbf{x}) = -\langle \boldsymbol{\mu} \rangle \cdot \mathbf{B}(\mathbf{x}),$$

where

$$\langle \boldsymbol{\mu} \rangle = \frac{e}{m}\left[F_1(0) + F_2(0)\right]\xi'^{\dagger}\frac{\boldsymbol{\sigma}}{2}\xi.$$

This expression for the magnetic moment of the electron can be rewritten in the standard form

$$\boldsymbol{\mu} = g\left(\frac{e}{2m}\right)\mathbf{S},$$

where $\mathbf{S}$ is the electron spin. The coefficient g, called the *Landé g-factor*, is

$$g = 2\left[F_1(0) + F_2(0)\right] = 2 + 2F_2(0). \tag{6.37}$$

Since the leading order of perturbation theory gives no F_2 term, QED predicts $g = 2 + \mathcal{O}(\alpha)$. The leading term is the standard prediction of the Dirac equation. In higher orders, however, we will find a nonzero F_2 and thus a small difference between the electron's magnetic moment and the Dirac value. We will compute the order-α contribution to this *anomalous magnetic moment* in the next section.

Section 6.3. Evaluate the lowest order vertex correction.