Section 6.3 : The electron vertex function

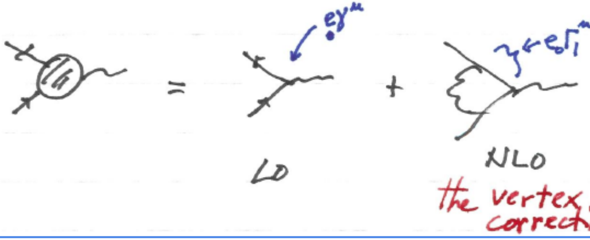
Electron scattering

$\mathcal{M} =$ 

← scattering center is a massive (static) charge

$q^\mu = p'^\mu - p^\mu$

$=$ momentum transfer



+ other corrections

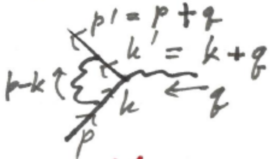
The part of $\mathcal{M}$ that involves the electron field

$$\bar{u}(p')\, e_0 \Gamma^\mu(p,p')\, u(p) = \bar{u}(p')\, e_0 \gamma^\mu\, u(p)$$

$$+ \ \bar{u}(p')\, e_0 \Gamma_1^\mu\, u(p) \ + \ .....$$

$$e_0 \Gamma_1^\mu = \int \frac{d^4k}{(2\pi)^4} \frac{g_{\rho\sigma}}{(p-k)^2 - \mu^2 + i\epsilon}$$

$$e_0 \gamma^\rho\, S_F(k)\, e_0 \gamma^\mu\, S_F'(k')\, e_0 \gamma^\sigma$$



$$\Gamma_1^{\mu} = e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^{\rho}(\not{k}+m)\gamma^{\mu}(\not{k}'+m)\gamma_{\rho}}{[(p-k)^2-\mu^2+i\epsilon](k^2-m^2+i\epsilon)(k'^2-m^2+i\epsilon)}$$

Use the Feynman integral formula to combine the denominators ...

$$\frac{1}{ABC} = \int_0^1\int_0^1\int_0^1 \frac{dx\,dy\,dz\,\delta(x+y+z-1)\,(3-1)!}{(xA+yB+zC)^3}$$

where $A = k^2-m^2+i\epsilon$

$B = k'^2-m^2+i\epsilon = (k+q)^2-m^2+i\epsilon$

$C = (k-p)^2-\mu^2+i\epsilon$

The denominator

$$x(k^2-m^2) + y[(k+q)^2-m^2] + z[(k-p)^2-\mu^2] + i\epsilon$$

$x+y+z=1$

$$= k^2(x+y+z) + 2k\cdot(yq-zp)$$
$$+ m^2(-x-y+z) + yq^2 - z\mu^2 \quad + i\epsilon$$

$$= k^2 + 2k\cdot Q + (2z-1)m^2 + yq^2 - z\mu^2$$

where $Q^{\mu} = yq^{\mu} - zp^{\mu}$

complete the square

$$= (k+Q)^2 - Q^2 + (2z-1)\,m^2 + y q^2 - z\mu^2$$

$$= -y^2 q^2 - z^2 m^2 + 2yz\, p\cdot q$$

note $p^2 = m^2$; $p'^2 = m^2$

$$q^2 = (p'-p)^2 = 2m^2 - 2p\cdot p'$$

$$2p\cdot q = 2p\cdot(p'-p) = 2p\cdot p' - 2m^2 = -q^2$$

$$= -y^2 q^2 - z^2 m^2 - yz\, q^2$$

$$= (k+Q)^2 - (1-z)^2 m^2 - z\mu^2$$

$$+ \left(y - y^2 - yz\right) q^2$$

$$y(1-y-z) = xy$$

$$= \ell^2 - \Delta + i\epsilon$$

where $\ell^\mu = k^\mu + Q^\mu = k^\mu + y\, q^\mu - z p^\mu$

and $\Delta = (1-z)^2 m^2 + z\mu^2 - xy\, q^2$

So $\Gamma_1^\mu = e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu}{(\ell^2 - \Delta + i\epsilon)^3} \; dx\,dy\,dz\, \delta(x+y+z-1)\, 2$

The numerator

$N^\mu = \gamma^\rho (\not{k} + m) \gamma^\mu (\not{k}' + m) \gamma_\rho$

$\leftarrow k' = k + q$

Now change the variable of integration from k to ℓ; $d^4k = d^4\ell$ because $k^\mu = \ell^\mu - Q^\mu$

$N^\mu = \gamma^\rho (\not{\ell} - \not{Q} + m) \gamma^\mu (\not{\ell} - \not{Q} + \not{q} + m) \gamma_\rho$

$\uparrow$ Simplify this using:

(i) $\gamma^\rho \gamma^\mu \gamma_\rho = -2\gamma^\mu$

$\gamma^\rho \gamma^\mu \gamma^\nu \gamma_\rho = 4 g^{\mu\nu}$

$\gamma^\rho \gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\rho = -2 \gamma^\delta \gamma^\beta \gamma^\alpha$

(ii) $\not{Q}\not{Q} = Q^2$ and similar

(iii) $\gamma^\mu \not{q} = 2q^\mu - \not{q}\gamma^\mu$ and similar

(iv) Make further simplifications for

$\bar{u}(p') N^\mu u(p)$

$\left\{ \begin{array}{l} \bar{u}(p') \not{p}\, u(p) = \bar{u}(p')\, m\, u(p) \\ \bar{u}(p') \not{q}\, u(p) = \bar{u}(p') (\not{p}' - \not{p}) u(p) \\ \quad = 0 \end{array} \right.$

Expand N^μ in powers of ℓ^μ

$$N^\mu = N_0^\mu + N_{1\ \alpha}^{\mu}\,\ell^\alpha + N_{2\ \alpha\beta}^{\mu}\,\ell^\alpha \ell^\beta$$

(that's all)

The quadratic term

$$N^\mu_{2\alpha\beta}\,\ell^\alpha \ell^\beta = \gamma^\rho \not{\ell}\, \gamma^\mu \not{\ell}\, \gamma_\rho$$

$$= \ell_\alpha \ell_\beta\, \gamma^\rho \gamma^\alpha \gamma^\mu \gamma^\beta \gamma_\rho$$

$$= \ell_\alpha \ell_\beta\,(-2)\,\gamma^\beta \gamma^\mu \gamma^\alpha$$

We will have $\displaystyle\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\alpha \ell^\beta}{(\ell^2 - \Delta + i\epsilon)^3}$

which by symmetry of Lorentz transformations is the same as obtained by the replacement

$$\ell^\alpha \ell^\beta = g^{\alpha\beta}\,\frac{1}{4}\,\ell^2$$

(sim. $x^i x^j \to \delta^{ij}\, r^2/3$ in 3D)

$$\therefore\ N_{2\ \alpha\beta}^{\mu}\,\ell^\alpha \ell^\beta \longrightarrow g_{\alpha'\beta'}\frac{\ell^2}{4}\,(-2)\,\gamma^{\beta'}\gamma^\mu \gamma^{\alpha'}$$

$$N_{2\ \alpha\beta}^{\mu} = g_{\alpha\beta}\, N_2^\mu \quad \text{where} \quad N_2^\mu = \frac{1}{4}(-2)\,\gamma^\rho \gamma^\mu \gamma_\rho$$

$$= -\frac{1}{2}(-2\gamma^\mu) = \gamma^\mu$$

The quadratic term is

$$e_0^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3} \bar{u}(p')\gamma^\mu u(p) \underbrace{dx\,dy\,dz\,\delta(x+y+z-1)\,2}_{\text{call this } D_{xyz}}$$

$$\Gamma^\mu_{1\,\text{quadratic}} = e_0^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2 \gamma^\mu}{(\ell^2-\Delta+i\epsilon)^3} D_{xyz}$$

$$e_0\,\Gamma_0^{\ \mu} + e_0\,\Gamma_{1\,\text{quadratic}}^{\ \mu} = \gamma^\mu e_0 (1 + e_0^2 L)$$

$$\text{where } L = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3} D_{xyz}$$

The linear term

$$N_1{}^\mu{}_\alpha \ell^\alpha = \gamma^\rho \not{\ell}\, \gamma^\mu (-\not{k}+\not{q}+m)\gamma_\rho + \gamma^\rho(-\not{k}+m)\gamma^\mu \not{\ell}\,\gamma_\rho$$

But this is irrelevant because we'll have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\alpha}{(\ell^2-\Delta+i\epsilon)^3} = 0$$

(The integrand is an odd function of ℓ^α)

The constant term (numerator indep. of ℓ)

$$N_0^\mu = \gamma^\rho(-\not{Q}+m)\gamma^\mu(-\not{Q}+\not{q}+m)\gamma_\rho$$

It takes some pages of algebra to simplify this. For example

$$\gamma^\rho(-\not{Q}+m)\gamma^\mu(-\not{Q}+m)\gamma_\rho$$

$$= m^2\gamma^\rho\gamma^\mu\gamma_\rho + m\gamma^\rho[-\not{Q}\gamma^\mu+\gamma^\mu(-\not{Q})]\gamma_\rho$$

$$+ \gamma^\rho\not{Q}\gamma^\mu\not{Q}\gamma_\rho$$

$$= -2m^2\gamma^\mu - mQ^\mu\cdot 4 - 2Q_\alpha Q_\beta\,\gamma^\beta\gamma^\mu\gamma^\alpha$$

$$= -2m^2\gamma^\mu - 4mQ^\mu - 2\not{Q}\underbrace{\gamma^\mu\not{Q}}_{2Q^\mu - \not{Q}\gamma^\mu}$$

$$= -2m^2\gamma^\mu - 4mQ^\mu - 4\not{Q}Q^\mu + 2Q^2\gamma^\mu$$

$$\bar{u}(p')\not{Q}u(p) = \bar{u}(p')[y\not{q} - z\not{p}]u(p)$$

$$= \bar{u}(p')(-zm)u(p)$$

$$\longrightarrow (2Q^2-2m^2)\gamma^\mu - 4mQ^\mu(1-z)$$

can be replaced by

$$\downarrow$$

$$(yq^\mu - zp^\mu)$$

After a lot of algebra we find

$$\bar{u}(p') N_0^\mu u(p) =$$

$$\bar{u}(p') \Big\{ \gamma^\mu \left[(1-x)(1-y) q^2 + (1-2z-z^2) m^2 \right]$$

$$+ (p^\mu + p'^\mu)\, m z (z-1)$$

$$+ q^\mu\, m (z-2)(x-y) \Big\}\, u(p)$$

The term $\propto q^\mu$ will give zero because

$$\int dx\, dy\, dz\, (x-y) = 0$$

$(x-y)$ changes sign if I exchange $x \leftrightarrow y$

Gordon's Identity

$$2m\, \bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[p^\mu + p'^\mu + i\sigma^{\mu\nu} q_\nu \right] u(p)$$

So we can replace

$$p^\mu + p'^\mu \rightarrow 2m\gamma^\mu - i\sigma^{\mu\nu} q_\nu$$

$$\bar{u}(p') N_0^\mu u(p)$$

$$= \bar{u}(p') \left\{ \gamma^\mu \left[(1-x)(1-y) q^2 + (1-4z+z^2) m^2 \right] + i\sigma^{\mu\nu} q_\nu \, m \, z(1-z) \right\} u(p)$$

Now we have

$$\Gamma_1^\mu = e_0^2 \int \frac{d^4 l}{(2\pi)^4} \frac{N_0^\mu}{(l^2 - \Delta + i\epsilon)^3} D_{xyz}$$

$$+ e_0^2 L \gamma^\mu$$

$$D_{xyz} = dx\,dy\,dz\; \delta(x+y+z-1) \cdot 2$$

where $L = \int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta + i\epsilon)^3} D_{xyz}$

Result so far…

… to be continued