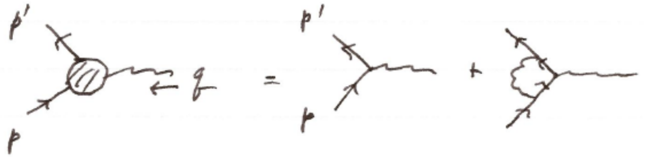
Section 6.3: The electron vertex function

Results so far ....



$$= \bar{u}(p')\; e_0 \Gamma^\mu \; u(p)$$

$$e_0 \Gamma^\mu = e_0 \Gamma_0^\mu + e_0 \Gamma_1^\mu + \cdots$$

$$= e_0 \gamma^\mu + e_0 \int \frac{d^4\ell}{(2\pi)^4} \frac{\hat{N}^\mu}{[\ell^2 - \Delta + i\epsilon]^3} D_{xyz}$$

where $D_{xyz} = 2\, dx\, dy\, dz\, \delta(x+y+z-1) \int_0^1$ (Feynman parameters)

$$\hat{N}_2^\mu = e_0^2 \gamma^\mu \frac{\ell^2}{4}$$ ← gives a divergent integral over ℓ^α. (UV divergence)

$$\hat{N}_0^\mu = e_0^2 \Big\{ \gamma^\mu \left[(1-x)(1-y)\, q^2 + (1-4z+z^2)\, m^2 \right] + i\sigma^{\mu\nu} q_\nu \left[m z (1-z) \right] \Big\}$$

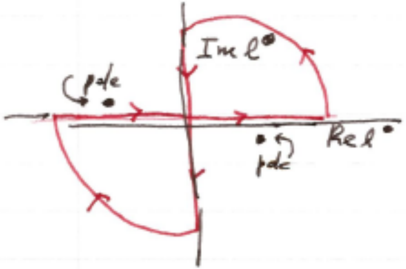
$$\Delta = (1-z)^2 m^2 + z\mu^2 - xy\, q^2$$

The convergent integral over ℓ^0

$$J_0 = \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{[\ell^2-\Delta+i\epsilon]^3}$$

Consider the complex ℓ^0 plane:

$$\int_{-\infty}^{\infty}\frac{d\ell^0}{[\ell^0-\sqrt{\vec{\ell}^2+\Delta}+i\epsilon][\ell^0+\sqrt{\vec{\ell}^2+\Delta}-i\epsilon]}$$



$$\oint = 0$$

$$= \int_{-\infty}^{\infty} + \underbrace{\int_{A_1}}_{=0} + \int_{+i\infty}^{-i\infty} + \underbrace{\int_{A_2}}_{=0}$$

$$\therefore \int_{-\infty}^{\infty} = \int_{-i\infty}^{i\infty}$$ "Wick Rotation"

Let $\ell^0 = i\ell'$

$d\ell^0 = i\,d\ell'$

$$\ell^2 = (\ell^0)^2 - \vec{\ell}^2 = -(\ell')^2 - \vec{\ell}^2 = -\ell_E^2$$

E: Euclidean

$$J_0 = \frac{-i}{(2\pi)^4}\int\frac{d^4\ell_E}{(\ell_E^2+\Delta)^3}$$

and now I can set $\epsilon = 0$.

Using polar coordinates in 4 dimensions:

$\{l_E^0, l_E^1, l_E^2, l_E^3\} =$

$\{r\cos\omega,\ r\sin\omega\sin\theta\cos\phi,\ r\sin\omega\sin\theta\sin\phi,$
$r\sin\omega\cos\theta\}$

$$d^4 l_E = r^3\,dr\ \sin^2\omega\,d\omega\ \sin\theta\,d\theta\ d\phi$$

$$J_0 = \frac{-i}{(2\pi)^4}\int_0^\infty \frac{r^3\,dr}{(r^2+\Delta)^3}\underbrace{\int_0^\pi \sin^2\omega\,d\omega}_{\pi/2}\cdot\underbrace{\int_0^\pi \sin\theta\,d\theta}_{2}\cdot\underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 2\pi^2$$

let $u = r^2$

$$J_0 = \frac{-i}{8\pi^2}\int_0^\infty \frac{u\,du/2}{(u+\Delta)^3}$$

$$J_0 = \frac{-i}{16\pi^2}\int_0^\infty du\left\{\frac{u+\Delta-\Delta}{(u+\Delta)^3}\right\}$$

$$= \frac{-i}{16\pi^2}\left\{\frac{-1}{u+\Delta} + \frac{\Delta/2}{(u+\Delta)^2}\right\}\Bigg|_{u=0}^{\infty}$$

$$= \frac{-i}{16\pi^2}\ \frac{1}{2\Delta} = \frac{-i}{32\pi^2\Delta}$$

$$e_0 \Gamma^{\mu}_{1,\text{convergent part}} = 2e_0^3 \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)$$

$$\frac{(-i)}{32\pi^2 \Delta} \left\{ \gamma^\mu \left[(1-x)(1-y) q^2 + (1-4z+z^2) m^2 \right] \right.$$

$$\left. + i\sigma^{\mu\nu} q_\nu \left[m\, z(1-z) \right] \right\}$$

<u>The divergent integral over ℓ^α</u>

Let $J_2 = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - \Delta + i\epsilon)^3}$

Important: It only comes in multiplying γ^μ, so it will be absorbed in RENORMALIZATION.

The UV divergence; logarithmically divergent;

This won't do!

We need <u>REGULARIZATION</u> to cut off the ℓ^μ integration at very very very large ℓ^μ.

- DIMENSIONAL REGULARIZATION: replace $d^4\ell$ by $d^n\ell$ where $n < 4$. (Seems crazy, but mathematically sound.)
- PAULI - VILLARS REGULARIZATION: Modify the photon propagator.

Replace $\dfrac{-g_{\rho\sigma}}{(p-k)^2+i\epsilon}$ (A)

by $-g_{\rho\sigma}\left\{ \dfrac{1}{(p-k)^2+i\epsilon} - \dfrac{1}{(p-k)^2-\Lambda^2+i\epsilon} \right\}$ (B)

where $\Lambda \to \infty$ at the end.

For $\Lambda = \infty$, (A) and (B) are equal.

For Λ finite, J_2 is finite.

As $\Lambda \to \infty$, J_2 will be divergent; perhaps that won't matter.

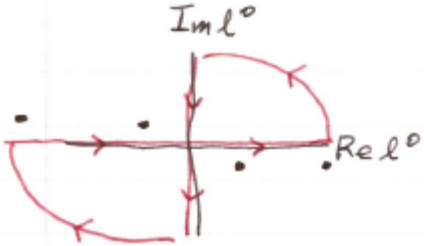
Evaluate the regularized integral

$$J_2 = \int \frac{d^4\ell}{(2\pi)^4} \left\{ \underset{\text{(divergent)}}{\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^2}} - \underset{\text{(divergent)}}{\frac{\ell^2}{(\ell^2-\Delta-z\Lambda^2+i\epsilon)^3}} \right\}$$

convergent

Recall $\Delta = (1-z)^2 m^2 + z\mu^2 - xy\, q^2$

Define $\Delta_\Lambda = \Delta + z\Lambda^2$



poles at

$\ell^0 = \pm\sqrt{\vec{\ell}^{\,2}+\Delta} \mp i\epsilon$ and

$\ell^0 = \pm\sqrt{\vec{\ell}^{\,2}+\Delta_\Lambda} \mp i\epsilon$

After Wick rotation

$$J_2 = \frac{-i}{(2\pi)^4}\int d^4\ell_E \left\{ \frac{-\ell_E^2}{(\ell_E^2+\Delta)^3} + \frac{\ell_E^2}{(\ell_E^2+\Delta_\Lambda)^3} \right\}$$

$$= \frac{i}{(2\pi)^4}\, 2\pi^2 \int_0^\infty r\,dr \left\{ \frac{r^4}{(r^2+\Delta)^3} - \frac{r^4}{(r^2+\Delta_\Lambda)^3} \right\}$$

$$= \frac{i}{16\pi^2}\int_0^\infty du \left\{ \frac{u^2}{(u+\Delta)^3} - \frac{u^2}{(u+\Delta_\Lambda)^3} \right\} \qquad u = r^2$$

$$= \frac{i}{16\pi^2}\left\{ \ln(u+\Delta) + \frac{2\Delta}{u+\Delta} - \frac{\Delta^2}{2(u+\Delta)^2} - \ln(u+\Delta_\Lambda) - \frac{2\Delta_\Lambda}{u+\Delta_\Lambda} + \frac{\Delta_\Lambda^2}{2(u+\Delta_\Lambda)^2} \right\}\Bigg|_0^\infty$$

$\{\ldots\} = 0$ at $u = \infty$

$\{\ldots\} = \ln\frac{\Delta}{\Delta_\Lambda}$ at $u = 0$

$$J_2 = \frac{-i}{16\pi^2}\ln\left(\frac{\Delta}{\Delta_\Lambda}\right) \quad \text{where} \left\{ \begin{array}{l} \Delta = (1-z)^2 m^2 + z\mu^2 - xy\,q^2 \\ \Delta_\Lambda = \Delta + z\Lambda^2 \end{array} \right.$$

$$\sim \frac{+i}{16\pi^2}\ln\frac{z\Lambda^2}{\Delta} + \underbrace{O(\Delta/\Lambda^2)}_{\text{(NEGLECT)}} \quad \text{as } \Lambda\to\infty$$

$$J_{0\,\text{REG.}} = J_0 + O(\Delta/\Lambda^2) \quad \text{as } \Lambda\to\infty$$

Renormalization and the anomalous magnetic moment

We have $\mathcal{M} = \bar{u}(p')\, e_0 \Gamma^\mu\, u(p)\, \hat{A}_\mu(q)$

The rest of the scattering amplitude

and by general symmetry principles (Section 6.2)

$$\mathcal{M} = \bar{u}(p')\left[\gamma^\mu e F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m} q_\nu\, e F_2(q^2)\right] u(p)\, \hat{A}_\mu(q)$$

The electron charge e is defined (experimentally) by low-energy scattering (e scattering or Compton scattering) ← Thomson cross section

← Rutherford cross section

The electron charge e is defined (experimentally) by low-energy $\underbrace{\text{electron scattering}}_{\text{Rutherford cross section}}$ or $\underbrace{\text{Compton scattering}}_{\text{Thomson scattering}}$.

For $\hat{A}_\mu(q) = \dfrac{\delta_{\mu 0}\, Ze}{(\vec{q})^2}$ (static charge Ze)

$$\mathcal{M} \xrightarrow[q\to 0]{} \bar{u}(p)\,\gamma^0 e F_1(0)\, u(p)\, \frac{Ze}{|\vec{q}|^2}$$ ← must give the Rutherford scattering c.s.

$\therefore\ e F_1(0) = e$ AN EXAMPLE OF RENORMALIZATION; $F_1(0) = 1$

But now consider scattering by a static magnetic field. $\vec{B} = \nabla \times \vec{A}$ (at low energies) $= \vec{B}(\vec{x})$

$$\hat{A}_\mu(q) = \hat{A}_i(\vec{q})\, \delta_{\mu i}\, \underbrace{\delta(q^0)}_{\text{static!}} \, (i = \Sigma_1^3)$$

and again $q^2 = 0$.

$$\mathcal{M} = \bar{u}(p')\Big[\gamma^\mu e \overset{q^2=0}{F_1(0)} + \frac{i\sigma^{\mu\nu}}{2m} q_\nu\, e \overset{q^2=0}{F_2(0)} \Big] u(p) \quad \hat{A}_i(\vec{q})\, \delta_{\mu i}$$

Gordon's identity

$$2m\, \bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\Big[\underbrace{p^\mu + p'^\mu}_{\to\, 0 \text{ for } \mu = i = 1,2,3 \text{ as } \vec{p} \to 0} + i\sigma^{\mu\nu} q_\nu \Big] u(p)$$

$p^\mu \to (m, \vec{0})$

$u(p) \to \begin{pmatrix} \binom{1}{0} \text{ or } \binom{0}{1} \\ \binom{0}{0} \end{pmatrix}$

For detailed discussion, P&S p.188

$$\mathcal{M} = \frac{ie}{2m}\big[F_1(0) + F_2(0)\big]\, \bar{u}(p)\, \sigma^{i\nu} q_\nu\, u(p)\, \hat{A}_i(\vec{q})\, \delta(q^0)$$

$$- \bar{u}(p)\, \sigma^{ij} q^j\, u(p)\, \hat{A}_i = -\bar{u}(p)\big[\vec{\sigma} \cdot \vec{q} \times \vec{\hat{A}}\big] u(p)$$

which is the cross section for (non relativistic because $\vec{p} \to 0$) scattering with the potential function $V(\vec{x}) = -\vec{\mu} \cdot \vec{B}(\vec{x})$ where we identify

$$\vec{\mu} = \frac{e}{m}\big[F_1(0) + F_2(0)\big]\, \xi^\dagger \frac{\vec{\sigma}}{2} \xi$$

Write $\vec{\mu} = g \left(\frac{e}{2m}\right) \vec{S}$

Note:
- $\frac{e}{2m} = \frac{e\hbar}{2mc}$ = Bohr magneton
- $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

Then

$$g = 2\big[F_1(0) + F_2(0)\big]$$
$$= 2 + 2F_2(0).$$

Now calculate $F_2(0)$

$$eF_2(0) = 2e_0^3 \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)$$
$$\frac{1}{32\pi^2\Delta}\left[m z(1-z)\right]$$

where $\Delta = (1-z)^2 m^2 + z\mu^2 - xy\,q^2 \longrightarrow (1-z)^2 m^2$ at $q^2=0$

(no IR divergence)
so $\mu^2 = 0$

Charge renormalization:

$$e = e_0\left(1 + Ke_0^2 + O(e_0^4)\right)$$

$$F_2(0) = \frac{e_0^3\, X}{e_0\left[1 + Ke_0^2 + O(e_0^4)\right]}$$
$$= e_0^2\, \underline{X} + O(e_0^4)$$

← higher order in perturbation theory; go to NNLO

$$= \frac{e^2\, \underline{X}}{1 + Ke_0^2} = e^2\, \underline{X} + \text{higher order}$$

$$F_2(0) = e^2\, \underline{X}$$ after renormalization, to the accuracy of perturbation theory that I have calculated

$$g = 2 + 2\,F_2(0)$$

$$F_2(0) = e^2\,X$$

The calculation of X is trivial

⊙ Apart from factors of 2 ....

$$X = \frac{1}{\pi^2}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\, \frac{m^2 z(1-z)}{(1-z)^2 m^2}$$

$$= \frac{1}{\pi^2}\int_0^1 \frac{dz}{1-z}\int_0^{1-z} dx$$

$$\boxed{\begin{array}{l} y = 1-z-x \\ y>0 \Rightarrow x<1-z \end{array}}$$

$$= 1/\pi^2$$

⊙ Restore the factors of 2 $\Rightarrow X = \dfrac{1}{8\pi^2}$

<u>Result</u>

$$g - 2 = 2F_2(0)$$

$$= \frac{e^2}{4\pi^2}$$

$$= \frac{\alpha}{\pi} \quad \text{where} \quad \alpha = \frac{1}{137}.$$

(Schwinger, 1948) shared Nobel prize with Feynman and Tomonaga,