The electron self-energy effect

The Dirac propagator in free field theory

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi$$

$$S_F(x-y) = \langle 0 | T\, \Psi(x)\bar{\Psi}(y) | 0 \rangle$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-y)}\, \hat{S}_F(p)$$

$$\hat{S}_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon}$$

which has poles at

$$p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2} \mp i\epsilon$$

Now consider Q.E.D.

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m_0)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e_0\,\bar{\Psi}\gamma_\mu\Psi A^\mu$$

e_0 : bare charge $\neq e$ the physical electron charge

m_0 : bare mass $\neq m$ the electron mass

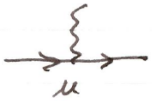
Rewrite the Lagrangian density as

$$\mathcal{L} = \bar{\Psi}\,(i\not{\partial} - m)\,\psi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$
$$+\;\bar{\Psi}\,\delta m\,\psi \;-\; e_0\,\bar{\Psi}\gamma_\mu\Psi A^\mu$$

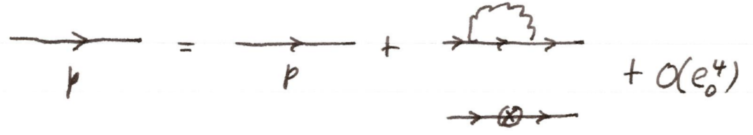
where $\delta m = m - m_0$

"mass renormalization"

and treat ...
the 1st line = the unperturbed theory;
the 2nd line = the perturbation theory

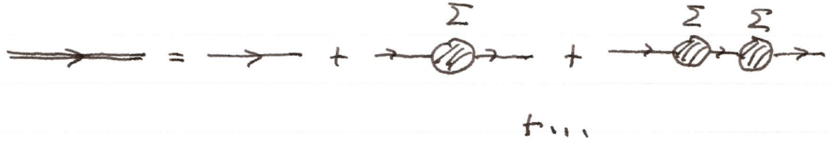
Feynman vertex  $= -ie_0\gamma_\mu$

and another vertex  $= i\,\delta m$



(We consider δm to be of order e_0^2.)

<u>Let Σ(p) be the sum of all one-particle irreducible diagrams</u>
Then



$$S = S_f + S_f\Sigma S_f + S_f\Sigma S_f\Sigma S_f + \cdots$$

$$S = \frac{1}{\not{p}-m} + \frac{1}{\not{p}-m}\Sigma\frac{1}{\not{p}-m} + \frac{1}{\not{p}-m}\Sigma\frac{1}{\not{p}-m}\Sigma\frac{1}{\not{p}-m} + \cdots$$

The electron propagator

Theorem

$$\frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \dots = \frac{1}{A-B}$$

Proof

$$(A-B)\left(\frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \frac{1}{A} B \frac{1}{A} B \frac{1}{A} + \dots\right)$$

$$= 1 + B\frac{1}{A} + B\frac{1}{A}B\frac{1}{A} + \dots$$

$$- B\frac{1}{A} - B\frac{1}{A}B\frac{1}{A} - B\frac{1}{A}B\frac{1}{A}B\frac{1}{A} - \dots$$

$$= 1 \qquad \text{Q.E.D.}$$

Apply the Theorem to S', with

$A = \not{p} - m$ and $B = \Sigma$ $\Rightarrow$

$$S' = \frac{1}{\not{p} - m - \Sigma(p) + i\epsilon}$$

By symmetry, with respect to Lorentz transformations, $\Sigma(p)$ must have this form:

$$\Sigma(p) = \alpha + \beta(\not{p}-m) + \Sigma_c(p)(\not{p}-m)$$

where α, β are constants and $\Sigma_c(p^2 = m^2) = 0$.

Now

$$S = \left[\not{p}-m - \alpha - \beta(\not{p}-m) - \Sigma_c(p)(\not{p}-m)\right]^{-1}$$

$$= \left[-\alpha + (\not{p}-m)(1-\beta) - \Sigma_c(p)(\not{p}-m)\right]^{-1}$$

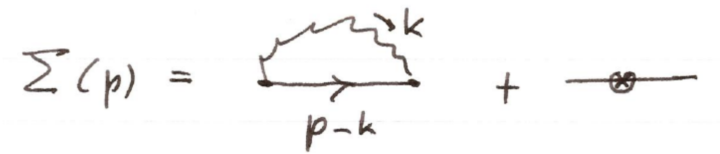
Also, S must have poles at

$p^0 = \pm\sqrt{\vec{p}^2+m^2} \mp i\epsilon$

$S^{-1}(p) = 0$ at $\not{p}=m \Rightarrow \alpha = 0$.

First-order perturbation theory for Σ (p)

The Feynman diagrams are



These give the function

$$= e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{k^2+i\epsilon} \frac{\gamma^\mu(\not p - \not k + m)\gamma^\nu}{(p-k)^2 - m^2 + i\epsilon} + \delta m$$

NB: This is m, not m_0!

Comment: Factors of i are ignored. Restore later.

Evaluation of Σ(p)

- Combine the denominators using the Feynman integral formula.
- Complete the square for the momentum in the denominator, and change the variable of integration from km to lm.
- Wick rotation
- Regularization (we'll use Pauli Villars regularization)
- Write Σ(p) in the form given earlier, i.e., in terms of aa bb and Σc(p).

$$\text{loop} = e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2(\not{p}-\not{k})+4m}{(k^2+i\epsilon)[(p-k)^2-m^2+i\epsilon]}$$

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax+b(1-x)]^2}$$

$$= e_0^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-2(\not{p}-\not{k})+4m}{[k^2x+(p-k)^2(1-x)-m^2(1-x)+i\epsilon]^2}$$

$$\text{denominator} = k^2x + (p^2-2p\cdot k+k^2)(1-x) - m^2(1-x)$$

$$= [k-p(1-x)]^2 - p^2(1-x)^2 + p^2(1-x) - m^2(1-x)$$

$$= \ell^2 + p^2x(1-x) - m^2(1-x)$$

where $\ell^\mu = k^\mu - p^\mu(1-x)$

Also, let $-\Delta = p^2x(1-x) - m^2(1-x)$

$$= e_0^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-2\not{p} + 2\not{\ell} + 2\not{p}(1-x) + 4m}{[\ell^2 + -\Delta + i\epsilon]^2}$$

Odd integrand : $\not{\ell} \to 0$

numerator $= -2\not{p}x + 4m$

Wick rotation

$$\text{loop} = e_0^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \frac{-2\not{p}x+4m}{[\ell_E^2+\Delta]^2}$$

Pauli Villars regularization $(\Lambda \to \infty)$

$$\frac{1}{(\ell_E^2+\Delta)^2} \to \frac{1}{(\ell_E^2+\Delta)^2} - \frac{1}{\ell_E^2+\Delta+\Lambda^2x}$$

$$\int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{(\ell_E^2+\Delta)^2} = \frac{2\pi^2}{(2\pi)^4} \int_0^\infty \frac{r^3\,dr}{(r^2+\Delta)^2}$$

$$= \frac{1}{16\pi^2} \int_0^\infty \frac{u\,du}{(u+\Delta)^2}$$

$$= \frac{1}{16\pi^2} \left\{ \ln(u+\Delta) - \Delta \frac{(-1)}{u+\Delta} \right\}_{u=0}^{\infty}$$

$$\text{loop} = \frac{e_0^2}{16\pi^2} \int_0^1 dx \, (-2\not{p}x+4m) \left\{ \ln\frac{u+\Delta}{u+\Delta+\Lambda^2x} + \frac{\Delta}{u+\Delta} - \frac{\Delta+\Lambda^2x}{u+\Delta+\Lambda^2x} \right\}_{u=0}^{\infty}$$

$$= \frac{e_0^2}{16\pi^2} \int_0^1 dx \, (-2\not{p}x+4m) \ln\frac{\Lambda^2x+\Delta}{\Delta}$$

where $-\Delta = p^2x(1-x) - m^2(1-x)$

Separate the divergent and convergent parts

$$\ln \frac{\Lambda^2 x + \Delta}{\Delta} = \ln \left(\frac{\Lambda^2}{m^2}\right)\left(\frac{m^2(\Lambda^2 x + \Delta)}{\Lambda^2 \Delta}\right)$$

$$= \ln \frac{\Lambda^2}{m^2} + \ln \frac{m^2 x}{\Delta} + \text{terms that } \to 0 \text{ as } \Lambda \to \infty$$

$\uparrow$ divergent as $\Lambda \to \infty$

$\uparrow$ convergent as $\Lambda \to \infty$

$$\text{loop} = \frac{e_0^2}{16\pi^2} \ln \frac{\Lambda^2}{m^2} \int_0^1 dx \, (-2\not{p}x + 4m) + C$$

$$= \underbrace{\qquad\qquad\qquad}_{G} (-\not{p} + 4m) + C$$

$$= 3mG - G(\not{p} - m) + C$$

The mass renormalization

$$\Sigma^{(2)}(p) = \underbrace{\delta m + 3mG}_{\alpha} \underbrace{- G(\not{p} - m)}_{\beta} + \mathcal{C}$$

- We must have $\alpha = 0$, so

$$\delta m = -3mG \quad ; \quad \text{or rather}$$

$$\delta m = 3mG = \frac{3e_0^2}{16\pi^2} m \ln \frac{\Lambda^2}{m^2}$$

(restoring factors of i that I dropped)

The mass renormalization is UV divergent, but only logarithmically divergent.

The propagator

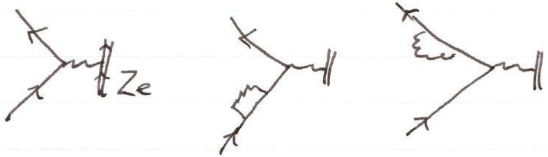
- $$S = \frac{1}{\not{p} - m + \beta(\not{p} - m) + \mathcal{C}}$$

$$= \frac{1}{\left[1 + \frac{e_0^2}{16\pi^2} \ln \frac{\Lambda^2}{m^2}\right](\not{p} - m) + \mathcal{C}}$$

This has a divergent factor! But there is another charge renormalization.

The charge renormalization

The charge renormalization from the electron self energy diagram.



Matrix element ~

$$S_f\, e_0\gamma^\mu S_F + S_F\, \Sigma^{(2)} S_F\, e_0\gamma^\mu S_f$$
$$+ S_f\, e_0\gamma^\mu S_f\, \Sigma^{(2)} S_F \quad + O(e_0^5)$$

$$\sim e_0 + e_0^3 \frac{1}{8\pi^2}\ln\frac{\Lambda^2}{m^2} \Longrightarrow e$$

$$e = Z_2 e_0 \quad \text{when} \quad Z_2 = 1 + \frac{e_0^2}{8\pi^2}\ln\frac{\Lambda^2}{m^2}$$

So once again, renormalization absorbs the UV divergences.

The Lamb shift

$$S = \frac{1}{\not p - m + \beta(\not p - m) + C}$$

$$= \frac{1}{\left[1 + \frac{e_0^2}{16\pi^2}\ln\frac{\Lambda^2}{m^2}\right](\not p - m) + C}$$

$$=$$

$$\frac{1}{\not p - m} + \frac{-\beta}{\not p - m} + \frac{-C}{(\not p - m)\,(\not p - m)}$$

Bethe showed how this change in the electron dynamics affects the energy levels of hydrogen