# Peskin and Schroedinger See their "Conventions and Notations"

#### Chapter 3: The Dirac Field

## The Dirac Equation

### /1/ Recall the Schroedinger equation

it 
$$\frac{2V}{\delta t} = -\frac{\hbar^2}{2m} \nabla^2 V$$
 $\frac{2V}{\delta t} = \frac{-1}{2m} \nabla^2 V$ 

The place wave solution

 $V(\vec{x},t) = C e^{i}(\vec{p}\cdot\vec{x} - Et)$ 
 $i(-iE)V = \frac{-1}{2m}(i\vec{p})^2 V$ 
 $V(\vec{x},t) = C e^{i}(\vec{p}\cdot\vec{x} - Et)$ 
 $V(\vec{x},t) = C e^{i}(\vec{x},t)$ 
 $V(\vec{x},t) = C e^{i}(\vec{x$ 

#### /2/ The Dirac equation

We want an equation that is (i) linear in time, (ii) with plane wave solutions, (iii) such that  $E = \sqrt{[p^2+m^2]}$ .

To be consistent with relativity, t and (x, y, z) should be treated similarly; because the Lorentz transformations mix t and (x, y, z). So let's try

$$i\frac{\partial \Psi}{\partial E} = (\vec{\alpha}.\vec{p} + \beta m) \Psi$$

$$E\Psi = (\vec{\alpha}.\vec{p} + \beta m) \Psi$$

$$E u(\vec{p}) = (\vec{\alpha}.\vec{p} + \beta m) u(\vec{p})$$

The quantities  $\beta$  and ( $\alpha_x \alpha_y \alpha_z$ ) will be *matrices*.

$$(\vec{\alpha} \cdot \vec{p} + \beta m) = u$$

$$= E (\vec{\alpha} \cdot \vec{p} + \beta m) u$$

$$= E^{2}u$$

$$= (\vec{\alpha} \cdot \vec{p} + \beta m)^{2}u$$

$$= \{\alpha'_{1}\alpha'_{1}\beta'_{1}\beta'_{2}\beta'_{1}\beta'_{2}\beta'_{1}\beta'_{2}\beta'_{1}\beta'_{2}\beta'_$$

# $\beta$ and ( $\alpha_x \alpha_y \alpha_z$ ) Since they don't commute, they must be matrices.

#### Four - vector notations

Define 
$$y^0 = \beta$$
;  
also,  $(y^1, y^2, y^3) = (\beta \alpha_x, \beta \alpha_y, \beta \alpha_z)$ 

### **UPPER AND LOWER INDICES:**

i 24 = -i2. 74 + Bm4

st be 
$$i \left( 8^{\circ} \frac{\partial \mathcal{V}}{\partial x^{\circ}} = -i \vec{y} \cdot \nabla \mathcal{V} + m \mathcal{V} \right)$$

This is the Dirac equation. Various notations may be used

$$i \ 8^{\mu} \frac{\partial 4}{\partial x^{\mu}} - m \ 4 = 0$$

$$i \ 8 \cdot \partial 4 - m \ 4 = 0$$

$$i \ 8 \cdot \partial 4 - m \ 4 = 0$$

$$defines: \ \beta = 8^{\mu} Q_{\mu}$$

#### /3/ The gamma matrices

What are the gamma matrices? *They are not unique.* 

The gamma matrices are 4x4 matrices, defined by certain anticommutation relations:

$$\{x^{i}, x^{j}\} = 2\delta_{ij}$$
  
 $\{x^{i}, y^{j}\} = \{\beta\alpha^{i}, \beta\alpha^{j}\} = -2\delta_{ij}$   
 $\{x^{i}, y^{o}\} = \{\beta\alpha^{i}, \beta\} = 0$   
 $\{x^{o}, y^{o}\} = 2(y^{o})^{2} = 2$ 

Thus, the defining equations are

$$\{ \gamma^{\mu}, \gamma^{\nu} \} = 2 g^{\mu\nu}$$

# The standard representation (Dirac) for the gamma matrices is

$$y^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } y^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix}$$

$$\sigma^{i} : Pauli matrices ; 1 = \begin{pmatrix} 10 \\ 01 \end{pmatrix}$$

Exercise. Verify (1).

# The chiral representation (Weyl) for the gamma matrices, which is used by P&S, is

It's not a very convenient choice.

Theorem. If  $\{\gamma^{\mu}, \gamma^{\nu}\} = 2 g^{\mu\nu}$ , and U is a unitary matrix ( $U^{\dagger}U = 1$ ), then  $\{ \gamma^{\prime \mu}, \gamma^{\prime \nu} \} = 2 g^{\mu \nu} \text{ where } \gamma^{\prime \mu} = U \gamma^{\mu} U^{\dagger}$ { } y'm, y'r} = Uymutuzvut Exerc Let  $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-1 \\ 1 \end{pmatrix}$   $1 = \begin{pmatrix} 61 \\ 61 \end{pmatrix}$ Show & = UyuUt

For most calculations, we don't need to use any specific representation of the gamma matrices. Instead we can use some identities that are true for all representations.

/4/ Examples of gamma matrix identities

# #. Trace $(\gamma^{\mu} \gamma^{\nu})$

Lemma. Trace(BA) = Trace(AB).

#### Proof.

Trace(BA) =  $B_{rs} A_{rs} = A_{rs} B_{rs} = Trace(AB)$ . Even if A and B do not commute, i.e.,  $BA \neq AB$ , always tr(BA) = tr(AB).

```
Trace (\gamma^{\mu} \gamma^{\nu})
        Tr xuxx
        = 1 Tr (8my + 8 x xm)
        = 1 Tr 2gm = 1 2gm Tr1
        = 42 m
#. Trace (\gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma})
      Tr (8mg x x x x x o)
      = 29 mv. 4 geo Tr(8v {84, 88}80)
                     + Tr (8 88 800)
                               5x401-xx4
      = 89 mg g - 8 gus qua
          +8 g a g rs - Tr (8 ps y 5 m)
      Tr(848 8880) = 4 guyso - 4 g g vo + 49 g v
```

$$\#$$
.  $\gamma^{\mu} \gamma^{\rho} \gamma_{\mu}$ 

$$y^{n} y^{p} y_{n} = \{ y^{n}, y^{p} \} y_{n} - y^{p} y_{n}^{y}$$
  
=  $2g^{ng} y_{n} - y^{p} \cdot 4$   
=  $-2y^{p}$ 

Etc.

We'll use a bunch of these identities. See the Appendix A.3.

#### /5/ The Dirac spinors

► The Dirac equation and the plane wave solutions:

$$(i \delta - m) \psi = 0$$

$$\psi(\bar{x}, t) = e^{i'(\bar{p} \cdot \bar{x} - Et)} u(\bar{p}, x)$$

$$e^{-ip \cdot x} \text{ where } p \cdot x = p^{i} x^{i} - \bar{p} \cdot \bar{x}$$

$$= p_{u} \times u$$

$$(\bar{p} - m) u = 0$$

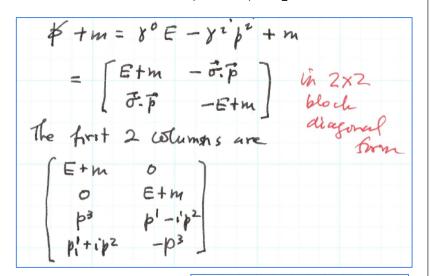
Now 
$$(p-m)(p+m)$$

$$= pp - m^2$$

$$= p^2 - m^2$$

$$= 0$$
So  $u(p, \mathfrak{F}) cm$  be any column
of  $p+m$ .

► In the standard (Dirac) representation:



$$u(\vec{p}, 1) = N\begin{pmatrix} E+m \\ 0 \\ p^{3} \\ p^{1}+ip^{2} \end{pmatrix}$$

$$u(\vec{p}, 2) = N\begin{pmatrix} 0 \\ E+m \\ p^{1}-ip^{2} \\ -p^{3} \end{pmatrix}$$

Normalization choice

We'll choose 
$$\overline{u} u = 2m$$
.

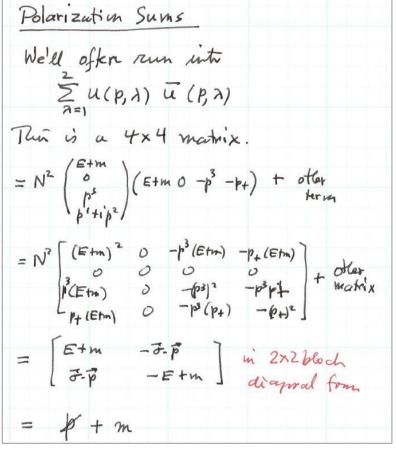
where  $\overline{u} = u^{\dagger} s^{\circ}$ .

 $\overline{u}(p_{1}) u(p_{1}) = N^{2}(6+m \circ p^{3} p_{-}) \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$ 

$$= N^{2}(E+m \circ p^{3} p_{-}) \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -p^{3} & -p^{2} & 0 \end{pmatrix}$$

$$= N^{2}(E+m \circ p^{3} p_{-}) \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -p^{3} & -p^{2} & 0 \end{pmatrix}$$

$$= N^{2}(E+m \circ p^{3})^{2} - p_{-} p_{+}$$



We derived this from the Dirac representation; the same result holds for any representation of the gamma matrices.

► Dirac spinors in the chiral (Weyl) representation of gamma matrices...

$$V_{c}^{M} = U y u U^{\dagger}$$

$$U_{c} = U u$$
For example,
$$U_{c}(p, 1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 10 & -10 & | & E+m \\ 0 & 1 & 0 & | & P^{3} \\ 10 & 0 & 1 & | & P^{3} \\ 0 & 0 & 1 & | & P^{3} \end{pmatrix} \xrightarrow{E+m} ILC Special Gase in PS:$$

$$V_{c} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} E+m & -p^{3} \\ E+m & +p^{3} \\ E+m & +p^{3} \end{pmatrix}$$

$$V_{c} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{E+m} & -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} & +\sqrt{E-m} \end{pmatrix}$$

$$U_{c} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{E+m} & -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} & +\sqrt{E-m} \end{pmatrix}$$

Dirac spinors for antiparticles:

Homework Set #1

The special case in PS:  

$$p' = p^2 = 0$$
;  $p_3 = \sqrt{E^2 - m^2}$   
 $u_c = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{E+m} - \sqrt{E-m}}{\sqrt{E+m} + \sqrt{E-m}} \right)$   
 $= \left( \frac{\sqrt{E-p^3}}{\sqrt{E+p^3}} \right)$   
 $= \frac{\sqrt{E-p^3}}{\sqrt{E+p^3}}$