Peskin and Schroedinger
See their "Conventions and Notations"

$\hbar = 1$ and $c = 1$

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$x^\mu = (x^0, \vec{x})$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$$

$$p \cdot x = g_{\mu\nu} p^\mu x^\nu = p^0 x^0 - \vec{p} \cdot \vec{x}$$

$$p^2 = p \cdot p = E^2 - |\vec{p}|^2 = m^2$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla \right)$$

$$\frac{e^2}{4\pi\hbar c} = \frac{1}{137}$$

Chapter 3 : The Dirac Field

The Dirac Equation
/1/ Recall the Schroedinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

$$i \frac{\partial \psi}{\partial t} = \frac{-1}{2m} \nabla^2 \psi \qquad (\hbar = 1)$$

The plane wave solution

$$\psi(\vec{x}, t) = C e^{i(\vec{p}\cdot\vec{x} - Et)}$$

$$i(-iE)\psi = \frac{-1}{2m} (i\vec{p})^2 \psi$$

$$E = p^2/2m$$

↳ a nonrelativistic approximation for particle energy

Also, $\vec{P} = -i\hbar\nabla = -i\nabla \qquad (\hbar = 1)$

So

$$\vec{P}\psi = -i(i\vec{p})\psi = \vec{p}\psi$$

The plane wave is an eigenstate of momentum.

/2/ The Dirac equation

We want an equation that is (i) linear in time, (ii) with plane wave solutions, (iii) such that E = √ [p^2+m^2].

$$\psi(\vec{x},t) = e^{i(\vec{p}\cdot\vec{x} - Et)}\, u(\vec{p})$$

$$\vec{P}\psi = -i\nabla\psi = \vec{p}\,\psi$$

$$i\frac{\partial\psi}{\partial t} = E\psi \leftarrow \sqrt{p^2+m^2}\,\psi$$

what about

$$i\frac{\partial\psi}{\partial t} = \sqrt{-\nabla^2+m^2}\,\psi \quad ?$$

To be consistent with relativity, ***t*** and ***(x, y,z)*** should be treated similarly; because the Lorentz transformations mix ***t*** and ***(x,y,z).***

So let's try

$$\vec{P} = -i\nabla$$

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{P} + \beta m)\,\psi$$

$$E\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi$$

$$E\,u(\vec{p}) = (\vec{\alpha}\cdot\vec{p} + \beta m)\,u(\vec{p})$$

The quantities β and (α_x α_y α_z) will be ***matrices***.

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\, E\, u$$
$$= E\,(\vec{\alpha}\cdot\vec{p} + \beta m)\, u$$
$$= E^2 u$$
$$= (\vec{\alpha}\cdot\vec{p} + \beta m)^2\, u$$
$$= \{\alpha_i \alpha_j p^i p^j + \beta^2 m^2 + m p^i (\alpha^i \beta + \beta \alpha^i)\}\, u$$
$$= (\vec{p}^{\,2} + m^2)\, u$$

So we must have

$$\tfrac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) = \delta_{ij}$$
$$\beta \alpha_i + \alpha_i \beta = 0$$
$$\beta^2 = 1.$$

β and (α_x α_y α_z)

Since they don't commute, they must be matrices.

Four - vector notations

Define $\gamma^0 = \beta$;

also, $(\gamma^1, \gamma^2, \gamma^3) = (\beta\alpha_x, \beta\alpha_Y, \beta\alpha_Z)$

UPPER AND LOWER INDICES:

$\{x^0, x^1, x^2, x^3\} = \{ct, x, y, z\}$ (c=1)

$\{x_0, x_1, x_2, x_3\} = \{ct, -x, -y, -z\}$

$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\} = \beta\{1, \alpha_x, \alpha_y, \alpha_z\}$

$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} = \beta\{1, -\alpha_x, -\alpha_y, -\alpha_z\}$

$\gamma \bullet A = \gamma^\mu \bullet A_\mu = \gamma_\mu \bullet A^\mu = \gamma^0 A^0 - \gamma^i A^i$

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\nabla\psi + \beta m\psi$$
$$i\gamma^0 \frac{\partial\psi}{\partial x^0} = -i\vec{\gamma}\cdot\nabla\psi + m\psi$$
$$i\left(\gamma^0 \frac{\partial}{\partial x^0} + \gamma^i \frac{\partial}{\partial x^i}\right)\psi - m\psi = 0$$

This is the Dirac equation.
Various notations may be used

$$i\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - m\psi = 0$$

$$i\gamma \cdot \partial \psi - m\psi = 0$$

$$i\not{\partial}\psi - m\psi = 0$$

defines : $\not{a} = \gamma^\mu a_\mu$

/3/ The gamma matrices

What are the gamma matrices?
They are not unique.

The gamma matrices are 4x4 matrices, defined by certain anticommutation relations:

$$\{\alpha^i, \alpha^j\} = 2\delta_{ij}$$

$$\{\gamma^i, \gamma^j\} = \{\beta\alpha^i, \beta\alpha^j\} = -2\delta_{ij}$$

$$\{\gamma^i, \gamma^0\} = \{\beta\alpha^i, \beta\} = 0$$

$$\{\gamma^0, \gamma^0\} = 2(\gamma^0)^2 = 2$$

Thus, the defining equations are

$$\{ \gamma^\mu , \gamma^\nu \} = 2\, g^{\mu\nu} \qquad (1)$$

The standard representation (Dirac) for the gamma matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

σ^i : Pauli matrices ; $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise. Verify (1).

The chiral representation (Weyl) for the gamma matrices, which is used by P&S, is

$$\gamma_c^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma_c^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

It's not a very convenient choice.

<u>Theorem.</u> If $\{ \gamma^\mu , \gamma^\nu \} = 2 g^{\mu\nu}$, and U is a unitary matrix ($U^\dagger U = 1$), then $\{ \gamma'^\mu , \gamma'^\nu \} = 2 g^{\mu\nu}$ where $\gamma'^\mu = U \gamma^\mu U^\dagger$.

$$\{\gamma'^\mu, \gamma'^\nu\} = U\gamma^\mu U^\dagger U \gamma^\nu U^\dagger + U\gamma^\nu U^\dagger U \gamma^\mu U^\dagger$$

$$= U \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2g^{\mu\nu} \mathbb{1}_4} U^\dagger = 2g^{\mu\nu}$$

<u>Exerc</u> Let $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Show $\gamma_c^\mu = U \gamma^\mu U^\dagger$

For most calculations, we don't need to use any specific representation of the gamma matrices. Instead we can use some identities that are true for all representations.

/4/ Examples of gamma matrix identities

#. Trace ($\gamma^\mu \gamma^\nu$)

Lemma. Trace(BA) = Trace(AB).

Proof.
Trace(BA) = $B_{rs} A_{rs} = A_{rs} B_{rs}$ = Trace(AB).
Even if A and B do not commute,
i.e., $BA \neq AB$, always tr(BA) = tr(AB).

#. Trace ($\gamma^\mu \gamma^\nu$)

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu$$
$$= \tfrac{1}{2} \mathrm{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$
$$= \tfrac{1}{2} \mathrm{Tr}\ 2 g^{\mu\nu} \quad = \tfrac{1}{2}\, 2 g^{\mu\nu}\, \mathrm{Tr}\, 1$$
$$= 4 g^{\mu\nu}$$

#. Trace ($\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$)

$$\mathrm{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma)$$
$$= \mathrm{Tr} (\{\gamma^\mu, \gamma^\nu\} \gamma^\rho \gamma^\sigma) - \mathrm{Tr} (\gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma)$$
$$= 2 g^{\mu\nu} \cdot 4 g^{\rho\sigma} - \mathrm{Tr} (\gamma^\nu \{\gamma^\mu, \gamma^\rho\} \gamma^\sigma)$$
$$+ \mathrm{Tr} (\gamma^\nu \gamma^\rho \underbrace{\gamma^\mu \gamma^\sigma}_{\{\gamma^\mu, \gamma^\sigma\} - \gamma^\sigma \gamma^\mu})$$
$$= 8 g^{\mu\nu} g^{\rho\sigma} - 8 g^{\mu\rho} g^{\nu\sigma} + 8 g^{\mu\sigma} g^{\nu\rho} - \mathrm{Tr} (\gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu)$$
$$\therefore$$
$$\mathrm{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4 g^{\mu\nu} g^{\rho\sigma} - 4 g^{\mu\rho} g^{\nu\sigma} + 4 g^{\mu\sigma} g^{\nu\rho}$$

#. $\gamma^\mu \gamma^\rho \gamma_\mu$

$$\gamma^\mu \gamma^\rho \gamma_\mu = \{\gamma^\mu, \gamma^\rho\}\gamma_\mu - \gamma^\rho \gamma^\mu \gamma_\mu$$
$$= 2g^{\mu\rho}\gamma_\mu - \gamma^\rho \cdot 4$$
$$= -2\gamma^\rho$$

Etc.

We'll use a bunch of these identities.
See the Appendix A.3.

/5/ The Dirac spinors

- The Dirac equation and the plane wave solutions:

$$(i\not{\partial} - m)\psi = 0$$

$$\psi_\alpha(\vec{x},t) = \underbrace{e^{i(\vec{p}\cdot\vec{x} - Et)}}_{e^{-ip\cdot x}}\, u_\alpha(\vec{p},\lambda)$$

where $p\cdot x = p^0 x^0 - \vec{p}\cdot\vec{x} = p_\mu x^\mu$

$$[i\gamma^\mu(-ip_\mu) - m]\psi = 0$$

$$(\not{p} - m)u = 0$$

Now $(\not{p} - m)(\not{p} + m)$

$$= \not{p}\not{p} - m^2$$

$$= p^2 - m^2$$

$$= 0$$

So $u(p,\lambda)$ can be any column of $\not{p} + m$.

- In the standard (Dirac) representation:

$$\not{p} + m = \gamma^0 E - \gamma^i p^i + m$$

$$= \begin{bmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{bmatrix}$$

in 2×2 block diagonal form

The first 2 columns are

$$\begin{bmatrix} E+m & 0 \\ 0 & E+m \\ p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix}$$

$$u(\vec{p},1) = N\begin{pmatrix} E+m \\ 0 \\ p^3 \\ p^1+ip^2 \end{pmatrix}$$

$$u(\vec{p},2) = N\begin{pmatrix} 0 \\ E+m \\ p^1-ip^2 \\ -p^3 \end{pmatrix}$$

Normalization choice

We'll choose $\bar{u}\,u = 2m$.

where $\bar{u} = u^\dagger \gamma^0$.

$$\bar{u}(p1)\,u(p1) = N^2 \begin{pmatrix} E+m & 0 & p^3 & p_- \end{pmatrix} \left(\begin{array}{cc|cc} 1 & 0 & & 0 \\ 0 & 1 & & \\ \hline & 0 & -1 & 0 \\ & & 0 & -1 \end{array}\right) \begin{pmatrix} E+m \\ 0 \\ p^3 \\ p_+ \end{pmatrix}$$

$$= N^2 \begin{pmatrix} E+m & 0 & p^3 & p_- \end{pmatrix} \begin{pmatrix} E+m \\ 0 \\ -p^3 \\ -p_+ \end{pmatrix}$$

$$= N^2 \left((E+m)^2 - (p^3)^2 - p_- p_+ \right)$$

$$= N^2 \left(E^2 + m^2 + 2Em - \vec{p}^{\,2} \right)$$

$$= N^2 \cdot 2m \cdot (E+m) = 2m$$

$$N = 1/\sqrt{E+m}$$

Polarization Sums

We'll often run into

$$\sum_{\lambda=1}^{2} u(p,\lambda)\,\bar{u}(p,\lambda)$$

This is a 4×4 matrix.

$$= N^2 \begin{pmatrix} E+m \\ 0 \\ p^3 \\ p^1 + i p^2 \end{pmatrix} \begin{pmatrix} E+m & 0 & -p^3 & -p_+ \end{pmatrix} + \text{other term}$$

$$= N^2 \begin{bmatrix} (E+m)^2 & 0 & -p^3(E+m) & -p_+(E+m) \\ 0 & 0 & 0 & 0 \\ p^3(E+m) & 0 & -(p^3)^2 & -p^3 p_+ \\ p_+(E+m) & 0 & -p^3(p_+) & -(p_+)^2 \end{bmatrix} + \text{other matrix}$$

$$= \begin{bmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{bmatrix} \quad \text{in } 2\times 2 \text{ block diagonal form}$$

$$= \not{p} + m$$

We derived this from the Dirac representation; the same result holds for any representation of the gamma matrices.

- Dirac spinors in the chiral (Weyl) representation of gamma matrices...

$$\gamma_c^\mu = U \gamma^\mu U^\dagger$$

$$\therefore\ u_c = U u$$

For example,

$$u_c(p,1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} E+m \\ 0 \\ p^3 \\ p^+ \end{pmatrix} \frac{1}{\sqrt{E+m}}$$

$$= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} E+m-p^3 \\ -p^+ \\ E+m+p^3 \\ p^+ \end{pmatrix}$$

The special case in PS:

$$p^1 = p^2 = 0\ ; \quad \therefore\ p_3 = \sqrt{E^2-m^2}$$

$$u_c = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{E+m} - \sqrt{E-m} \\ 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E-p^3} \\ 0 \\ \sqrt{E+p^3} \\ 0 \end{pmatrix}$$

Eq. (3.52)

Dirac spinors for antiparticles:

Homework Set #1