Quantization of the Dirac Field (Sec 3.5)

$$(i\,\gamma.\partial - m)\,\psi = 0$$

We can expand $\psi(x)$ in plane wave solutions because they are complete...

$$\boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left\{ a(ps)\, u(ps)\, e^{-ip\cdot x} + b^\dagger(ps)\, v(ps)\, e^{ip\cdot x} \right\}}$$

important factor

Here $p^0 = E_p = +\sqrt{\vec{p}^2 + m^2}$

PS quotation, page 58:
"All the expressions we will need in our later work are listed below; corresponding expressions above, where they differ, should be forgotten."

Spinor definitions

$(\gamma.p - m)\, u(p,s) = 0$ $\quad$ $(s=1,2)$

$(\gamma.p + m)\, v(p,s) = 0$ $\quad$ (")

$\bar{u}\, u = 2\,m$ and $\bar{v}\, v = -2\,m$ where $\bar{u} = u^+ \gamma^0$.

Then also

$$\boxed{\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left\{ a^\dagger(ps)\, \bar{u}(ps)\, e^{ip\cdot x} + b(ps)\, \bar{v}(ps)\, e^{-ip\cdot x} \right\}}$$

The coefficients a(**p**,s) and b(**p**,s) become annihilation operators.

Comment on the spin-statistics theorem

/1/ Canonical quantization

$$L = \int \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \; d^3x$$

The canonical momentum conjugate to ψ

$$\pi = \frac{\delta L}{\delta \dot{\psi}} = \bar{\psi} i\gamma^0 = i\psi^+$$

So the Dirac quantization condition is

$$\{ \psi(\vec{x}t), \pi(\vec{y}t) \} = i\,\delta^3(\vec{x}-\vec{y})$$

or $\{ \psi(x,t), \psi^+(\vec{y}t) \} = \delta^3(\vec{x}-\vec{y})$

Recall Lagrange's equation

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\psi}} - \frac{\delta L}{\delta \psi} = 0$$

$$\frac{\partial}{\partial t} \left(\bar{\psi} i\gamma^0 \right) - \left(\bar{\psi} \, (-i\vec{\gamma}\cdot\overleftarrow{\nabla}) - m\bar{\psi} \right) = 0$$

$$\bar{\psi} \left(i\gamma^0 \overleftarrow{\partial}_0 + i\vec{\gamma}\cdot\overleftarrow{\nabla} + m \right) = 0$$

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{\bar{\psi}}} - \frac{\delta L}{\delta \bar{\psi}} = 0 \qquad (3.35)$$

$$0 - \left(i\gamma^\mu \partial_\mu - m \right) \psi = 0$$

$$\left(i\gamma\cdot\partial - m \right) \psi = 0 \qquad (3.34)$$

/2/ Second quantization

(familiar from PHY 855; or read P&S)

$$\{a(ps), a^\dagger(p's')\} = (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')\delta_{ss'}$$
$$\{b(ps), b^\dagger(p's')\} = (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')\delta_{ss'}$$

<u>all</u> other anticommutators involving $a, a^\dagger, b, b^\dagger$ are 0.

/3/ The equal time anticommutation relations (E.T.aC.R.)

$$\{\psi_\alpha(x), \psi_\beta(y)\} \text{ at } t_x = t_y$$
$$\propto \{a \text{ and } b^\dagger, a \text{ and } b^\dagger\}$$
$$= 0$$

$$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} \text{ at } t_x = t_y$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\sum_{s,s'}$$
$$\{a\, u_\alpha e^{-ip\cdot x} + b^\dagger v_\alpha e^{ip\cdot x},$$
$$a^\dagger u_\beta^\dagger e^{iq\cdot y} + b\, v_\beta^\dagger e^{-iq\cdot y}\}$$

Note: $(2\pi)^3\delta^3(p-q)\delta_{ss'}$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s\{e^{-ip\cdot(x-y)}u_\alpha(ps)u_\beta^\dagger(ps)$$
$$+ e^{ip\cdot(x-y)}v_\alpha(ps)v_\beta^\dagger(ps)\}$$

Note: $x^0 = y^0$; in the second line change $\vec{p}$ to $-\vec{p}$.; $p^0 = E_p$

$$\sum_s u\bar{u} = \not{p} + m$$
$$\sum_s v\bar{v} = \not{p} - m$$

$$\not{p} \to \gamma^0 E_p - \vec{\gamma}\cdot\vec{p}$$

$$= \int\frac{d^3p}{(2\pi)^3}e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}\Big\{\frac{1}{2E_p}(\not{p}+m)\gamma^0$$
$$+ \frac{1}{2E_p}(\gamma^0E_p + \vec{\gamma}\cdot\vec{p} - m)\gamma^0\Big\}_{\alpha\beta}$$
$$= \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$
$$= \delta^3(\vec{x}-\vec{y}) \quad \text{which is canonical!}$$

/4/ The Feynman propagator = $S_F(x-y)$

Recall from PHY 855 -- we want the propagator for the *time-ordered product of fields*. (Do you remember why?)

Here is the definition of $S_F(x-y)$:

$$S_F(x-y) = \langle 0 | T\, \psi(x)\, \bar{\psi}(y) | 0 \rangle = \begin{cases} \langle 0 | \psi(x)\, \bar{\psi}(y) | 0 \rangle & \text{if } x^0 > y^0 \\ -\langle 0 | \bar{\psi}(y)\, \psi(x) | 0 \rangle & \text{if } x^0 < y^0 \end{cases}$$

Here is the formula for $S_F(x-y)$, as a Fourier integral:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

$\epsilon \to 0^+$ is implied

P&S call $S_F(x-y)$ "*the Green's function with Feynman boundary conditions*".

/5/ Derivation of the Fourier integral for S_F

First, calculate $\langle 0 | \psi(x)\, \bar{\psi}(y) | 0 \rangle$ from second quantization

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{s'} \langle 0 | \left(a u e^{-ip\cdot x} + b^\dagger v e^{ip\cdot x} \right) \left(a^\dagger \bar{u} e^{iq\cdot y} + b \bar{v} e^{-iq\cdot y} \right) | 0 \rangle$$

$b|0\rangle = 0$ and $\langle 0 | b^\dagger = 0$

$\langle 0 | a_p a_q^\dagger | 0 \rangle = (2\pi)^3 \delta^3(p-q)\, \delta_{ss'}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_s u\bar{u}}_{\not{p}+m} e^{-ip\cdot(x-y)}$$

Similarly

$$\langle 0 | \bar{\psi}(y)\, \psi(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{\not{p}-m}{2E_p} e^{ip\cdot(x-y)}$$

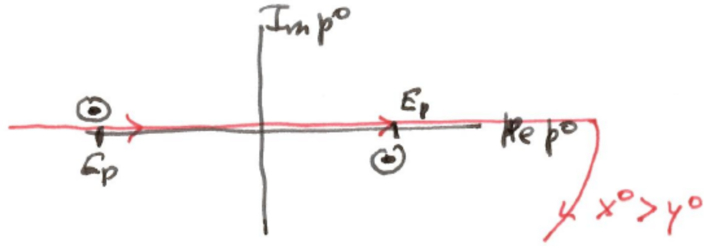
Second, calculate the Fourier integral over p^0

$$\mathcal{G} = \int \frac{dp^0}{2\pi} \int \frac{d^3p}{(2\pi)^3} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} \, e^{-ip^0(x^0-y^0)} \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

If $x^0 > y^0$ then $e^{-ip^0(x^0-y^0)} \to 0$ in the L.H. p^0 P.

$$e^{-i(-iR)(x^0-y^0)} = e^{-R(x^0-y^0)}$$

so close the contour below



Singularities:

$$p^2 - m^2 + i\epsilon = p^{0\,2} - \vec{p}^{\,2} - m^2 + i\epsilon$$

$$= p^{0\,2} - E_p^2 + i\epsilon$$

$$= (p^0 - E_p + i\epsilon)(p^0 + E_p - i\epsilon)$$

Note: $(E_p - i\epsilon)^2 = E_p^2 - 2i\epsilon E_p$
$= E_p^2 - i\epsilon$ because $\epsilon \to 0^+$

Poles at $p^0 = \pm E_p \mp i\epsilon$

Residue of the pole at $p^0 = E_p - i\epsilon$

$$\mathcal{G} = \frac{-2\pi i}{2\pi} \int \frac{i(\gamma^0 E_p - \vec{\gamma}\cdot\vec{p} + m)}{2E_p} \, e^{-iE_p(x^0-y^0)} \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{d^3p}{(2\pi)^3}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\not{p}+m}{2E_p} e^{-ip\cdot(x-y)} \quad \text{w/ } (p^0 = E_p)$$

which is correct for $x^0 > y^0$.

If $x^0 < y^0$ then close the contour above ; pole is at $p^0 = -E_p + i\epsilon$; residue theorem $\Rightarrow$

$$\mathcal{J} = \frac{2\pi i}{2\pi} \int \frac{i(-\gamma^0 E_p - \vec{\gamma}\cdot\vec{p} + m)}{(-2E_p)} \, e^{iE_p(x^0-y^0)} \, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{d^3p}{(2\pi)^3}$$

let $\vec{p} \to -\vec{p}$

$$\mathcal{J} = -\int \frac{d^3p}{(2\pi)^3} \frac{\gamma^0 E_p - \vec{\gamma}\cdot\vec{p} - m}{2E_p} e^{ip\cdot(x-y)}$$

$$= -\int \frac{d^3p}{(2\pi)^3} \frac{\not{p} - m}{2E_p} e^{ip\cdot(x-y)}$$

(here $p^0 = E_p$)

which is correct for $x^0 < y^0$.

QED

Homework Assignments 1 and 2 are both due Friday.