CHAPTER 5 : EXAMPLES IN QUANTUM ELECTRODYNAMICS

These are the processes that are calculated in chapter 5:

- $e^+ e^- \to \mu^+ \mu^-$

This is a basic example in QED. Also important because of the similar process $e^+ e^- \to q \bar{q}$, which proved the quark model and QCD by describing $q\bar{q} \to$ hadrons; PETRA (DESY) and PEP (SLAC) accelerator experiments.

- $e^- + \mu^- \to e^- + \mu^-$

An example of crossing symmetry.
Is scattering by a muon realistic? No, but the muon could be the projectile. also important because of the similar process $e^- + q \to e^- + q$, which occurs in electron-proton deep-inelastic scattering (ep DIS); SLAC and HERA (DESY) experiments.

- $\gamma e^- \to \gamma e^-$

Compton scattering.

- $e^+ e^- \to \gamma \gamma$

Pair annihilation into photons.

Other examples

- $e^- e^- \to e^- e^-$

Moller scattering. This is interesting as an example of identical particles.

- $e^+ e^- \to e^+ e^-$

Bhabha scattering. This process is used to determine the luminosity of $e^+ e^-$ collisions. Provides the best upper limit on the radius of the electron.

THE FEYNMAN RULES FOR QED

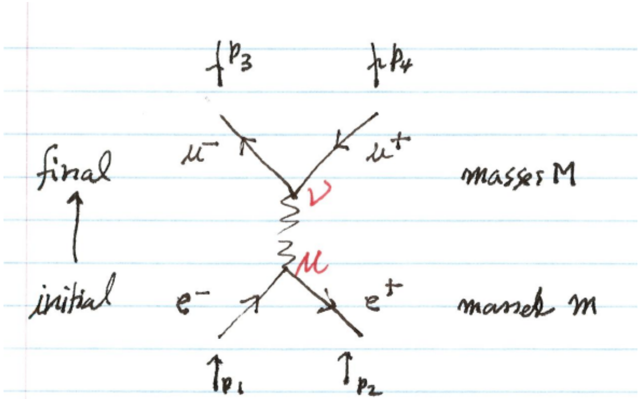
- draw the topologically distinct connected diagrams.
- vertex factor = e γ^{μ} ; 4-momentum is conserved at the vertex.
- fermion propagator = $S_F(p)$
- photon propagator = $D_{\mu\nu}(q)$
- external fermion leg = a Dirac spinor; $u(p,s)$, $\bar{u}(p,s)$, $v(p,s)$, $\bar{v}(p,s)$
- external photon leg = a polarization 4-vector; $\varepsilon_{\mu}(q)$
- verify the relative signs !

E E-bar → Mu Mu-bar

$$e^-(p_1) + e^+(p_2) \rightarrow \mu^-(p_3) + \mu^+(p_4)$$

$$p_1^{\mu} + p_2^{\mu} = p_3^{\mu} + p_4^{\mu}$$

There is only one Feynman diagram



Thus the transition matrix element is

$$\mathcal{M} = \bar{v}(p_2)\, e\gamma^{\mu}\, u(p_1)\; \bar{u}(p_3)\, e\gamma^{\nu}\, v(p_4)\; D_{\mu\nu}(p_1+p_2)$$

$$\mathcal{M} = \bar{v}(p_2)\, e\gamma^\mu\, u(p_1)\ \bar{u}(p_3)\, e\gamma^\nu\, v(p_4)\ D_{\mu\nu}(p_1+p_2)$$

$$D_{\mu\nu} = \frac{g_{\mu\nu}}{(p_1+p_2)^2} = \frac{g_{\mu\nu}}{s}$$

$$\mathcal{M} = \frac{e^2}{s}\left(\bar{v}_2 \gamma^\mu u_1\right)\left(\bar{u}_3 \gamma_\mu v_4\right)$$

The goal is to calculate the cross section. Recall the steps in the calculation.

☆ Square *M*

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\ \bar{v}_2 \gamma^\mu u_1\ \bar{u}_1 \gamma^\nu v_2\ \bar{u}_3 \gamma_\mu v_4\ \bar{v}_4 \gamma_\nu u_3$$

because

$$\gamma^0 (\gamma^\nu)^\dagger \gamma^0 = \begin{cases} \gamma^0\gamma^0\gamma^0 = \gamma^0 & \text{for } \nu = 0 \\ \gamma^0(-\gamma^i)\gamma^0 = \gamma^i & \text{for } \nu = i \end{cases}$$

$$= \gamma^\nu$$

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\left(\mathrm{Tr}\ \gamma^\mu u_1 \bar{u}_1 \gamma^\nu v_2 \bar{v}_2\right)\left(\mathrm{Tr}\ \gamma_\mu v_4 \bar{v}_4 \gamma_\nu u_3 \bar{u}_3\right)$$

☆ For unpolarized annihilation, average over s_1 and s_2 , and sum over s_3 and s_4 .

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{s^2}\,\frac{1}{4}\,\mathrm{Tr}\,\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_2 - m)\ \mathrm{Tr}\ \gamma_\mu(\not{p}_4 - M)\gamma_\nu(\not{p}_3 + M)$$

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2} E^{\mu\nu} M_{\mu\nu}$$

☆ Calculate the traces

Traces

$$E^{\mu\nu} = \mathrm{Tr}\, \gamma^\mu (\not p_1 + m) \gamma^\nu (\not p_2 - m)$$

$$= \mathrm{Tr}\, \gamma^\mu \not p_1 \gamma^\nu \not p_2 - m^2\, \mathrm{Tr}\, \gamma^\mu \gamma^\nu$$

$$= 4\, p_1^\mu p_2^\nu - 4 g^{\mu\nu}\, p_1 \cdot p_2 + 4 p_2^\mu p_1^\nu$$
$$- m^2\, 4 g^{\mu\nu}$$

$$= 4 \left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu \right)$$
$$- 4 g^{\mu\nu} \left(p_1 \cdot p_2 + m^2 \right)$$

$$M_{\mu\nu} = \mathrm{Tr}\, \gamma_\mu (\not p_4 - M) \gamma_\nu (\not p_3 + M)$$

$$= 4 \left(p_{4\mu} p_{3\nu} + p_{4\nu} p_{3\mu} \right)$$
$$- 4 g_{\mu\nu} \left(p_3 \cdot p_4 + M^2 \right)$$

$$E^{\mu\nu} M_{\mu\nu}$$

$$= 16 \left(p_1 \cdot p_4\, p_2 \cdot p_3 \times 2 + p_1 \cdot p_3\, p_2 \cdot p_4 \times 2 \right)$$
$$- 16 \left[p_1 \cdot p_2 \times 2 \left(p_3 \cdot p_4 + M^2 \right) \right]$$
$$- 16 \left[p_3 \cdot p_4 \times 2 \left(p_1 \cdot p_2 + m^2 \right) \right]$$
$$+ 16 \cdot 4 \left(p_1 \cdot p_2 + m^2 \right) \left(p_3 \cdot p_4 + M^2 \right)$$

To simplify, substitute the Mandelstam variables —

$$s = (p_1 + p_2)^2 = 2m^2 + 2 p_1 \cdot p_2$$
$$s = (p_3 + p_4)^2 = 2M^2 + 2 p_3 \cdot p_4$$

$$t = (p_1 - p_3)^2 = m^2 + M^2 - 2 p_1 \cdot p_3$$
$$t = (p_4 - p_2)^2 = m^2 + M^2 - 2 p_2 \cdot p_4$$

$$u = (p_1 - p_4)^2 = m^2 + M^2 - 2 p_1 \cdot p_4$$
$$u = (p_3 - p_2)^2 = m^2 + M^2 - 2 p_2 \cdot p_3$$

☆ Substitute Mandelstam variables (See page 156)

$$E^{\mu\nu}M_{\mu\nu}$$

$$= 8\,(m^2+M^2-u)^2 + 8\,(m^2+M^2-t)^2$$

$$-32\, p_1\cdot p_2\; s/2 \;-32\, p_3\cdot p_4\; s/2$$

$$+64\; s/2 \cdot s/2$$

$$= 8\,(m^2+M^2-u)^2 + 8\,(m^2+M^2-t)^2$$

$$+\,32\,(m^2+M^2)\, s/2$$

☆ Result

Result

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4s^2}\, E\cdot M$$

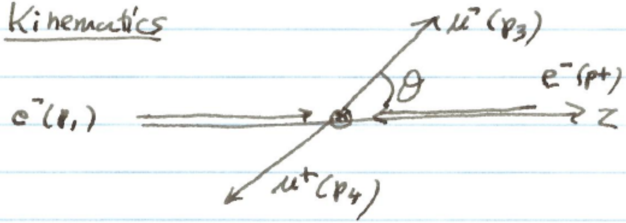
$$= \frac{e^4}{4s^2}\, 8\left\{ (m^2+M^2-u)^2 + (m^2+M^2-t)^2 + 2(m^2+M^2)s \right\}$$

$$= \frac{2e^4}{s^2}\left\{ (m^2+M^2-u)^2 + (m^2+M^2-t)^2 + 2(m^2+M^2)s \right\}$$

Note: This is Lorentz invariant and dimensionless.

☆ The center-of-mass differential cross section

Kinematics



$$p_1^\mu = (E_1,\ 0,\ 0,\ p_1)$$

$$p_2^\mu = (E_1,\ 0,\ 0,\ -p_1)$$

$$p_3^\mu = (E_1,\ p_3\sin\theta,\ 0,\ p_3\cos\theta)$$

$$p_4^\mu = (E_1,\ -p_3\cos\theta,\ 0,\ -p_3\cos\theta)$$

$$s = (p_1+p_2)^2 = 4E_1^2$$

$$t = (p_1-p_3)^2 = m^2+M^2-2E_1^2+2p_1p_3\cos\theta$$

$$u = (p_1-p_4)^2 = m^2+M^2-2E_1^2-2p_1p_3\cos\theta$$

The cross section

$$\sigma = \frac{1}{2E_1 2E_2 v_{rel}} \int \frac{d^3p_3}{(2\pi)^3\, 2E_3} \int \frac{d^3p_4}{(2\pi)^3\, 2E_4} |\mathcal{M}|^2$$

$$(2\pi)^4\, \delta(E_3+E_4-E_1-E_2)\, \delta^3(\vec{p}_3+\vec{p}_4)$$

(in the center of mass frame, all the particle energies are equal)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{(2\pi)^2} \frac{1}{(2E_1)^4} \frac{1}{v_{rel}} \int p_3^2\, dp_3\, \delta\left(2\sqrt{p_3^2+M^2} - 2\sqrt{p_1^2+m^2}\right)$$

$$\text{integral} = \frac{p_3^2}{|f'(p_3)|} \quad \text{where } f(p_3) = 2\sqrt{p_3^2+M^2} - 2\sqrt{p_1^2+m^2}$$

$$f'(p_3) = 2\,\tfrac{1}{2}(p_3^2+M^2)^{-1/2}\cdot 2p_3 = 2p_3/E_3 \quad (E_3 = E_1)$$

$$v_{rel} = |v_1 - v_2| = \frac{2p_1}{E_1}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{(2\pi)^2} \frac{1}{(2E_1)^4} \frac{E_1}{2p_1} \frac{p_3^2 E_1}{2p_3}$$

$$= \frac{|\mathcal{M}|^2}{(2\pi)^2} \frac{1}{64E_1^2} \frac{p_3}{p_1}$$

In terms of the energy E_1 = the energy of one of the initial electrons,

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\overline{|\mathcal{M}|^2}}{(2\pi)^2} \frac{1}{64E_1^2} \left(\frac{E_1^2-M^2}{E_1^2-m^2}\right)^{1/2}$$

Now write $\overline{|\mathcal{M}|^2}$ in terms of E_1 and θ:

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{s^2}\Big\{(m^2+M^2-u)^2 + (m^2+M^2-t)^2 + 2(m^2+M^2)s\Big\}$$

$$= \frac{2e^4}{s^2}\Big\{(2E_1^2 + 2p_1p_3\cos\theta)^2 + (2E_1^2 - 2p_1p_2\cos\theta)^2 + 2(m^2+M^2)\,4E_1^2\Big\}$$

Further simplifications

$$e^2 = 4\pi\alpha \quad \text{where} \quad \alpha = \frac{1}{137}$$

$$\sqrt{s} = E_{CM} = 2E_1$$

$$\overline{|\mathcal{M}|^2} = \frac{2(4\pi\alpha)^2}{s \cdot 4E_1^2}\Big\{ 2(2E_1^2)^2 + 2(2p_1p_3\cos\theta)^2 + 2(m^2+M^2)\cdot 4E_1^2 \Big\}$$

$$= \frac{(4\pi\alpha)^2}{s}\Big\{ 4E_1^2 + 4\left(\frac{p_1^2\,p_3^2}{E_1\,E_1}\right)\cos^2\theta + 4(m^2+M^2) \Big\}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\mathcal{M}|^2}{(2\pi)^2}\,\frac{p_3}{64E_1^2 p_1}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{4s}\Big\{ 1 + \frac{m^2+M^2}{E_1^2} + \left(1-\frac{m^2}{E_1^2}\right)\left(1-\frac{M^2}{E_1^2}\right)\cos^2\theta \Big\}\frac{p_3}{p_1}$$

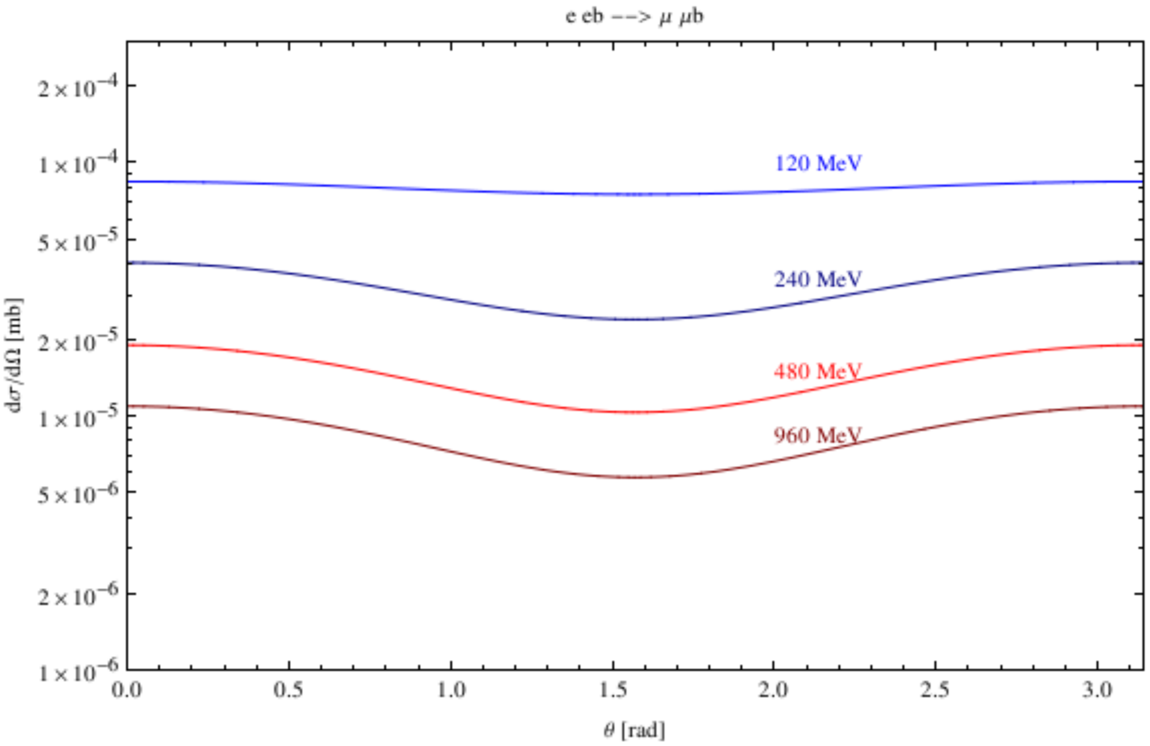
$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{4s}\left(\frac{E_1^2-M^2}{E_1^2-m^2}\right)^{1/2} \times \Big\{ 1 + \frac{m^2+M^2}{E_1^2} + \left(1-\frac{m^2}{E_1^2}\right)\left(1-\frac{M^2}{E_1^2}\right)\cos^2\theta \Big\}$$

$$s = E_{CM}^2 = 4E_1^2$$

We can approximate m = 0 because m << M and m << E. Then the result is PS equation 5.12.

The high-energy limit is

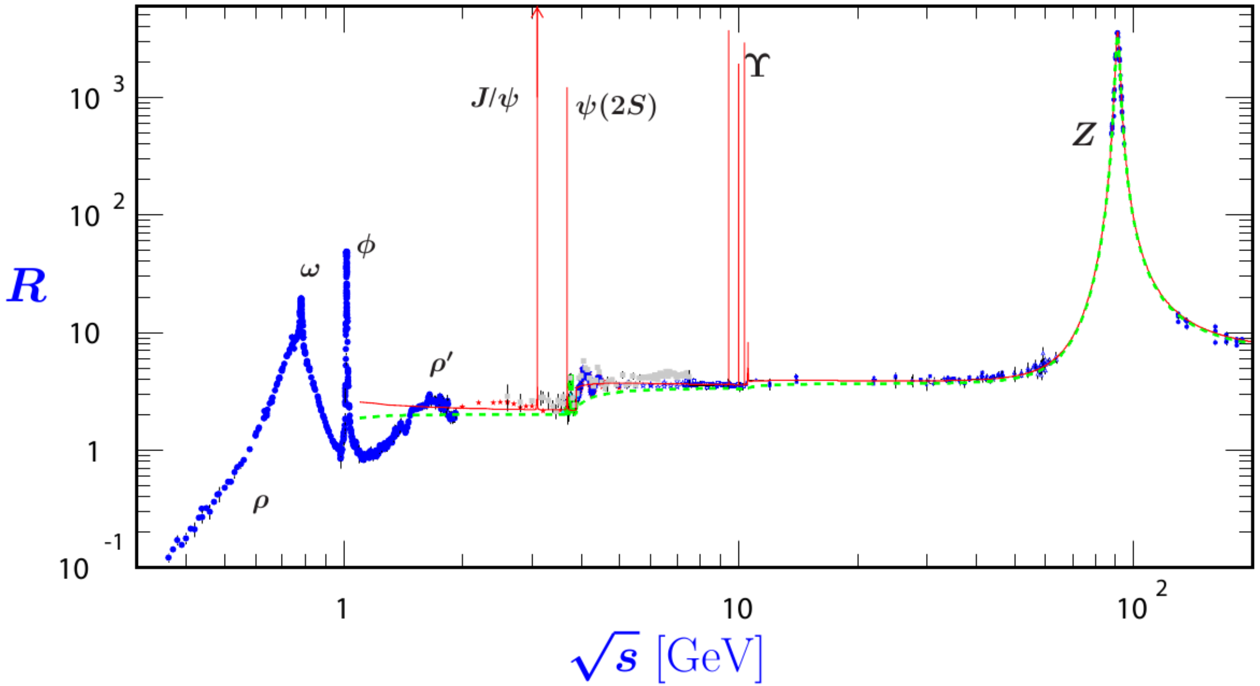
$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$



e eb --> μ μb
120 MeV
240 MeV
480 MeV
960 MeV
dσ/dΩ [mb]
θ [rad]

Electron - positron annihilation to hadrons

These were important experiments in the history of high-energy physics.
From the particle data group, the figure shows the ratio
R = σ (e e → hadrons) / σ (e ebar → μ μbar) .
The underlying process in hadron production is e- + e+ → q + qbar.
Neglecting QCD interactions we would just have R = constant (green = naive quark model; red = QCD 3rd order).



The √{s} dependence is due to 2 effects:

- thresholds
- resonances

But between thresholds we do indeed have R ≈ constant.

And we can even estimate the constant: above the b-quark threshold,

$R = \Sigma\, e_q^{\ 2} / e^2$

$= (1/9 + 4/9 + 1/9 + 4/9 + 1/9) \times 3$

$= 11 / 3 = 3.67$.

This is a great success of QCD.

Historically, measurements of R provided evidence for the quark model and QCD.