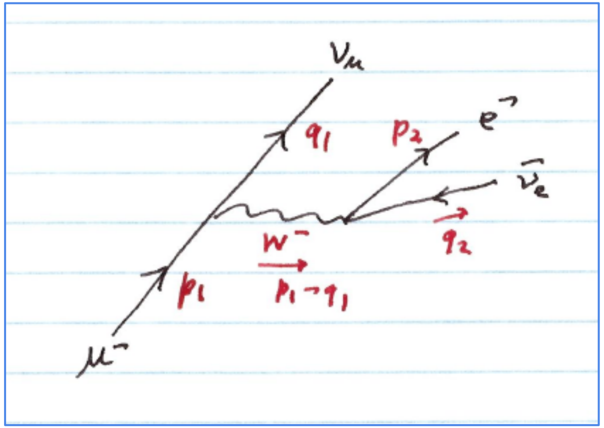
An example of the weak interaction - the muon decay rate

The mean lifetime of a muon in the rest frame is 2.2 μs. We'll calculate that from the Fermi weak coupling. The decay is

$$\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e$$

$$p_1^\mu = q_1^\mu + p_2^\mu + q_2^\mu$$

<u>The Feynman diagram</u>



$$\mathcal{M} = \bar{u}(q_1)\, g\gamma^\mu (1-\gamma_5)\, u(p_1)\; \bar{u}(p_2)\, g\gamma^\nu (1-\gamma_5)\, v(q_2)\; D_{\mu\nu}(p_1 - q_1)$$

The W-boson propagator is

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2}$$

But M_W = 80 GeV is much larger than the muon mass, so we can approximate

$$D_{\mu\nu}(k) = -g_{\mu\nu}/M_W^2$$

Also, we define the Fermi weak coupling G by

$$\frac{G}{\sqrt{2}} = \frac{g^2}{M_W^2}$$

$$\mathcal{M} = \frac{G}{\sqrt{2}} \; \bar{u}(q_1)\, \gamma^\mu (1-\gamma_5)\, u(p_1)$$
$$\bar{u}(p_2)\, \gamma_\mu\, (1-\gamma_5)\, v(q_2)$$

Calculate the square of *M*.

For example,

$$\bar{u}(q_1)\,\gamma^\mu(1-\gamma_5)\,u(p_1)\left[\bar{u}(q_1)\,\gamma^\nu(1-\gamma_5)\,u(p_1)\right]^\dagger$$
$$= \bar{u}(q_1)\,\gamma^\mu(1-\gamma_5)\,u(p_1)\; u^\dagger(p_1)\underbrace{(1-\gamma_5)^\dagger(\gamma^\nu)^\dagger(\gamma^0)^\dagger}\,u(q_1)$$

$$= \gamma^0\gamma^0\,(1-\gamma_5)\,(\gamma^\nu)^\dagger\,\gamma^0$$
$$= \gamma^0\gamma^0\,(\gamma^\nu)^\dagger\,\gamma^0\,(1-\gamma_5) \quad \text{why?}$$
$$= \gamma^0\,\gamma^\nu\,(1-\gamma_5) \quad \text{why?}$$

$$= \bar{u}(q_1)\,\gamma^\mu(1-\gamma_5)\,u(p_1)\,\bar{u}(p_1)\,\gamma^\nu(1-\gamma_5)\,u(q_1)$$
$$= \mathrm{Tr}\;\gamma^\mu(1-\gamma_5)\,u\bar{u}(p_1)\;\gamma^\nu(1-\gamma_5)\,u\bar{u}(q_1)$$

For an unpolarized muon, average over s_1.

Also, sum over $s_2\ s_3\ s_4$.

Then

$$\frac{1}{2}\sum u\bar{u}(p_1) = \frac{1}{2}(\not{p}_1 + M)$$

$$\sum u\bar{u}(q_1) = \not{q}_1 \quad (\nu_\mu \text{ mass} = 0)$$

Similarly for the electron part

$$\sum u\bar{u}(p_2) = \not{p}_2 + m_e$$

$$\sum v\bar{v}(q_2) = \not{q}_2 \quad (\nu_e \text{ mass} = 0)$$

Result (for unpolarized decay)

$$\overline{|\mathcal{M}|^2} = \frac{G^2}{2}\frac{1}{2}\,\mathrm{Tr}\,\gamma^\mu(1-\gamma_5)(\not{p}_1+M)\gamma^\nu(1-\gamma_5)\not{q}_1$$

$$\mathrm{Tr}\,\gamma_\mu(1-\gamma_5)\not{q}_2\gamma_\nu(1-\gamma_5)(\not{p}_2+m)$$

$$= \frac{G^2}{4} M^{\mu\nu} E_{\mu\nu}$$

Muon tensor (p_1^μ and q_1^μ)

$$M^{\mu\nu} = \mathrm{Tr}\,\gamma^\mu(1-\gamma_5)(\not{p}_1+M)\gamma^\nu(1-\gamma_5)\not{q}_1$$

$$\mathrm{Tr}\,\gamma\gamma\gamma = 0$$

$$= \mathrm{Tr}\,\gamma^\mu(1-\gamma_5)\underbrace{\not{p}_1\gamma^\nu(1-\gamma_5)}_{(1-\gamma_5)\not{p}_1\gamma^\nu}\not{q}_1$$

$$1+1-2\gamma_5 = 2(1-\gamma_5)$$

$$= 2\,\mathrm{Tr}(1+\gamma_5)\gamma^\mu\not{p}_1\gamma^\nu\not{q}_1$$

$$= 2(S^{\mu\nu} + A^{\mu\nu})$$

Symmetric part

$$S^{\mu\nu} = \mathrm{Tr}\, \gamma^\mu \not{p}_1 \gamma^\nu \not{q}_1$$

$$= p_{1\alpha} q_{1\beta}\, \mathrm{Tr}\, \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta$$

$$= p_{1\alpha} q_{1\beta} \left[4 g^{\mu\alpha} g^{\nu\beta} - 4 g^{\mu\nu} g^{\alpha\beta} + 4 g^{\mu\beta} g^{\nu\alpha} \right]$$

$$= 4 \left[p_1^\mu q_1^\nu + p_1^\nu q_1^\mu - g^{\mu\nu} p_1\cdot q_1 \right]$$

Antisymmetric part

$$A^{\mu\nu} = \mathrm{Tr}\, \gamma_5 \gamma^\mu \not{p}_1 \gamma^\nu \not{q}_1$$

$$= p_{1\alpha} q_{1\beta}\, \mathrm{Tr}\, \gamma^5 \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta$$

$$= p_{1\alpha} q_{1\beta}\, 4i\, \epsilon^{\mu\alpha\nu\beta}$$

$$M^{\mu\nu} = 2\left[S^{\mu\nu}(p_1, q_1) + A^{\mu\nu}(p_1, q_1) \right]$$

Now $\overline{|\mathcal{M}|^2} = \frac{G^2}{4} M^{\mu\nu} E_{\mu\nu}$

$$= G^2 \left[S^{\mu\nu}(p_1,q_1) + A^{\mu\nu}(p_1,q_1) \right] \left[S_{\mu\nu}(q_2,p_2) + A_{\mu\nu}(q_2,p_2) \right]$$

$$S^{(\mu)}\cdot S^{(e)} = 16\left(p_1^\mu q_1^\nu + p_1^\nu q_1^\mu - g^{\mu\nu} p_1\cdot q_1 \right)\left(q_{2\mu} p_{2\nu} + q_{2\nu} p_{2\mu} - g_{\mu\nu}\, q_2\cdot p_2 \right)$$

$$= 16\{ 2\, p_1\cdot q_2\; p_2\cdot q_1 + 2\, p_1\cdot p_2\; q_1\cdot q_2 - 2\, p_1\cdot q_1\; p_2\cdot q_2 - 2\, p_1\cdot q_1\; p_2\cdot q_2 + 4\, p_1\cdot q_1\; p_2\cdot q_2 \}$$

$$= 32\{ p_1\cdot p_2\; q_1\cdot q_2 + p_1\cdot q_2\; p_2\cdot q_1 \}$$

$$A^{(\mu)}\cdot A^{(e)} = (4i)^2\, p_{1\alpha} q_{1\beta}\, q_2^{\alpha'}\, p_2^{\beta'}\, \underbrace{\epsilon^{\mu\alpha\nu\beta}\, \epsilon_{\mu\alpha'\nu\beta'}}_{-2\delta^\alpha_{\alpha'}\delta^\beta_{\beta'} + 2\delta^\alpha_{\beta'}\delta^\beta_{\alpha'}}$$

$$= -32\{ -p_1\cdot q_2\; p_2\cdot q_1 + p_1\cdot p_2\; q_1\cdot q_2 \}$$

SIGN is CRITICAL!

Result

$$|M|^2 = 64\ G^2\ \ p_1.q_2\ \ p_2.q_1$$

The decay rate

Equation 4.86 (in rest frame of μ)

$$dR = \frac{1}{2M}\left(\prod_f \frac{d^3k_f}{(2\pi)^3\, 2E_{k_f}}\right)|M|^2$$

$$(2\pi)^4\ \delta^4\!\left(p_1 - \sum_f k_F\right)$$

(rate = $|T|^2$/time ;
analogous to the cross section formula)

The neutrinos are unobservable,
so we'll integrate over $\vec{q}_1$ and $\vec{q}_2$

$\Rightarrow$ differential rate w.r.t. electron momentum.

Phase space integral for the two neutrinos

The 3-body final state (μ rest frame)

$$\frac{1}{2M}\int\frac{d^3p_2}{(2\pi)^3\, 2E_{p_2}}\int\frac{d^3q_1}{(2\pi)^3\, 2q_1}\int\frac{d^3q_2}{(2\pi)^3\, 2q_2}\ |M|^2$$

$$(2\pi)^4\ \delta(M - E_e - q_1 - q_2)\ \delta^3(\vec{p}_2 + \vec{q}_1 + \vec{q}_2)$$

$$E_e = \sqrt{p_2^2 + m_e^2}$$

Let $q^\mu = p_1^\mu - p_2^\mu = q_1^\mu + q_2^\mu$

Rest frame: $q^0 = M - E_e$

$\vec{q} = \vec{q}_1 + \vec{q}_2 = -\vec{p}_2$

$$dR = \frac{1}{2M}\int \frac{d^3 p_2}{(2\pi)^3\, 2E_e}\; \frac{64\, G^2\, p_{1\alpha}\, p_{2\beta}}{(2\pi)^2\, 2\cdot 2}$$

$$\times \int \frac{d^3 q_1}{q_1}\,\frac{d^3 q_2}{q_2}\,\delta^4(q_1+q_2-q)\, q_2^\alpha\, q_1^\beta$$

call this $I^{\alpha\beta}(q)$; note that it is symmetric in $\alpha\beta$

Use some tricks to calculate $I^{\alpha\beta}(q)$

$$I^{\alpha\beta} = \int \frac{d^3 q_1}{|\vec{q}_1|}\,\frac{d^3 q_2}{|\vec{q}_2|}\, q_1^\alpha\, q_2^\beta\, \delta^4(q_1+q_2-q)$$

$I^{\alpha\beta}$ is a Lorentz tensor (L.I.P.S.) and it only depends on q^μ, which is a Lorentz vector. Therefore $I^{\alpha\beta}$ must have the form

$$I^{\alpha\beta}(q) = g^{\alpha\beta} A(q^2) + q^\alpha q^\beta B(q^2)$$

where A and B are scalars.

① Consider

$$g_{\alpha\beta} I^{\alpha\beta} = 4A + q^2 B$$

$$= \int \frac{d^3 q_1}{q_1} \frac{d^3 q_2}{q_2} \underbrace{q_1 \cdot q_2}_{} \, \delta^4(q_1 + q_2 - q)$$

$= \frac{1}{2}(q_1+q_2)^2$ because $q_1^2 = q_2^2 = 0$ (as 4 vectors)

$= \frac{1}{2} q^2$

$$= \frac{1}{2} q^2 \; I(q^2) \text{ where}$$

$$I(q^2) = \int \frac{d^3 q_1}{q_1} \frac{d^3 q_2}{q_2} \, \delta^4(q_1 + q_2 - q).$$

I is a scalar, so we may evaluate it in any Lorentz frame. Use the frame where $\vec{q} = 0$.

Then $q^2 = (q^0)^2$ and

$$I = \int \frac{d^3 q_1}{|\vec{q}_1|} \frac{1}{|-\vec{q}_1|} \delta(|\vec{q}_1| + |-\vec{q}_1| - q^0)$$

$$= 4\pi \int \frac{q_1^2 \, dq_1}{q_1^2} \delta(2q_1 - q^0)$$

$$= 2\pi.$$

$$4A + q^2 B = 2\pi \frac{q^2}{2} = \pi q^2$$

(2) Now consider

$$q_\alpha q_\beta I^{\alpha\beta} = q^2 A + (q^2)^2 B$$

$$= \int \frac{d^3 q_1}{q_1} \frac{d^3 q_2}{q_2} \; q\cdot q_1 \; q\cdot q_2 \; \delta^4(q_1+q_2-q)$$

$$q\cdot q_1 = (q_1+q_2)\cdot q_1 = q_2\cdot q_1 \quad \text{b/c } q_1^2 = 0$$

$$q\cdot q_2 = (q_1+q_2)\cdot q_2 = q_1\cdot q_2 \quad \text{"}$$

$$\therefore q\cdot q_1 \; q\cdot q_2 = (q_1\cdot q_2)^2 = \frac{1}{4}\left[(q_1+q_2)^2\right]^2$$

$$= \frac{1}{4}(q^2)^2$$

$$q^2 A + (q^2)^2 B = \frac{1}{4}(q^2)^2 \underbrace{2\pi}_{I \text{ again}}$$

$$A + q^2 B = \frac{\pi}{2} q^2$$

$$4A + q^2 B = \pi q^2$$

$$A = \frac{\pi}{6} q^2 \quad \text{and} \quad B = \frac{\pi}{3}$$

$$I^{\alpha\beta}(q) = \frac{\pi}{6}\left[g^{\alpha\beta} q^2 + 2 q^\alpha q^\beta \right]$$

The decay rate:

$$dR = \frac{1}{2M}\int \frac{d^3 p_2}{(2\pi)^3 2E_e} \frac{4G^2}{\pi^2} \underbrace{p_{1\alpha} p_{2\beta} I^{\alpha\beta}}$$

$$\frac{\pi}{6}\left[p_1\cdot p_2 \, q^2 + 2 \, p_1\cdot q \; p_2\cdot q \right]$$

where $q = p_1 - p_2$

$$= \frac{\pi}{6}\Big[p_1\cdot p_2 \left(M^2 + m^2 - 2p_1\cdot p_2\right)$$

$$+ 2\left(M^2 - p_1\cdot p_2\right)\left(p_1\cdot p_2 - m^2\right)\Big]$$

$$= \frac{\pi}{6}\left\{ 3\, p_1\cdot p_2 \left(M^2 + m^2\right) - 4(p_1\cdot p_2)^2 - 2M^2 m^2 \right\}$$

The decay rate:

$$dR = \frac{1}{2M}\int \frac{d^3p_2}{(2\pi)^3\, 2E_e}\, \frac{4G^2}{\pi^2}\, p_{1\alpha}p_{2\beta}\, I^{\alpha\beta}$$

$$\frac{\pi}{6}\left[\, p_1\cdot p_2\, q^2 + 2\, p_1\cdot q\ \, p_2\cdot q \,\right]$$

where $q = p_1 - p_2$

$$= \frac{\pi}{6}\Big[\, p_1\cdot p_2\,(M^2 + m^2 - 2p_1\cdot p_2) + 2\,(M^2 - p_1\cdot p_2)(p_1\cdot p_2 - m^2)\Big]$$

$$= \frac{\pi}{6}\left\{ 3\, p_1\cdot p_2\,(M^2+m^2) - 4(p_1\cdot p_2)^2 - 2M^2m^2 \right\}$$

In the muon rest frame, $\vec{p}_1 = 0$;

$p_1\cdot p_2 = ME_e$.

$$\frac{\pi}{6}\left\{ 3ME_e(M^2+m^2) - 4M^2E_e^2 - 2M^2m^2 \right\}$$

$$= \frac{\pi}{6}M\left\{ 3E_e(M^2+m^2) - 4ME_e^2 - 2Mm^2 \right\}$$

Collect all the factors.

Note $\int d^3p_2 = \int p_2^2\, dp_2\, d\Omega = 4\pi\, p_2^2\, dp_2$

$E_2^2 = p_2^2 + m^2 \Rightarrow \qquad = 4\pi\, p_2\, E_2\, dE_2$

So the rate, differential in electron energy, is

$$\frac{dR}{dE_e} = \frac{1}{2M}\, \frac{4\pi\, p_2\, E_e}{(2\pi)^3\cdot 2E_e}\, \frac{4G^2}{\pi^2}\, \frac{\pi M}{6}\left\{ 3E_e(M^2+m^2) - 4ME_e^2 - 2Mm^2 \right\}$$

$$\frac{dR}{dE_e} = \frac{G^2}{12\pi^3}\sqrt{E_e^2 - m^2}\,\Big\{ 3E_e(M^2+m^2) - 4ME_e^2 - 2Mm^2 \Big\}$$

$$\frac{dR}{dE_e} = \frac{G^2}{12\pi^3}\sqrt{E_e^2 - m^2}\left\{3E_e(M^2+m^2) - 4ME_e^2 - 2Mm^2\right\}$$

Muon decay rate

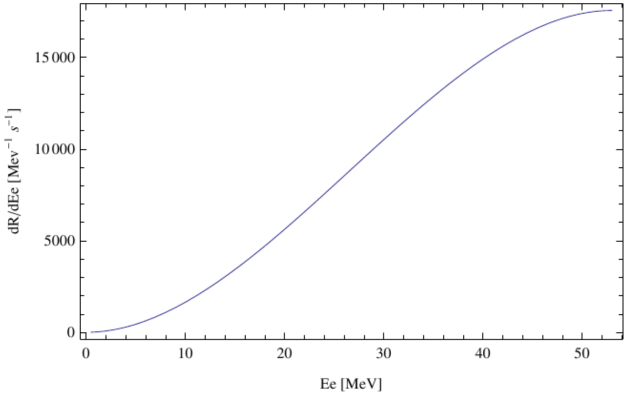
differential rate

```
In[66]:= GF = 1.166*^-5 * (UGeV) ^ -2;
         UGeV = 1000;
         hb = 197 / (2.998*^23);
         GF
Out[69]= 1.166 × 10^-11

In[57]:= dRdEe[Ee_, M_, me_] := (1 / hb) * GF ^ 2 / (12 * Pi ^ 3) * Sqrt[Ee ^ 2 - me ^ 2] *
           (3 * Ee * (M ^ 2 + me ^ 2) - 4 * M * Ee ^ 2 - 2 * M * me ^ 2)

In[58]:= Plot[dRdEe[Ee, 106, 1 / 2], {Ee, 1 / 2, 53},
          Frame → True, FrameLabel → {"Ee [MeV]", "dR/dEe [Mev^-1 s^-1]"}]
```

Out[58]=



homework

total rate