Compton Scattering

$$\gamma + e \rightarrow \gamma + e$$

$$k^\mu + p^\mu = k'^\mu + p'^\mu$$

Lab frame of reference : $p^\mu = (m, \vec{0})$

i.e. initial electron at rest ; $k^\mu = (\omega, \vec{k})$

$$\sigma = \frac{1}{2E_k \cdot 2E_p \cdot v_{rel}} \int \frac{d^3k'}{(2\pi)^3 \cdot 2E_{k'}} \int \frac{d^3p'}{(2\pi)^3 2E_{p'}}$$

$$|M|^2 \; (2\pi)^4 \; \delta^4(p' + k' - p - k)$$

~~In~~ The lab frame :

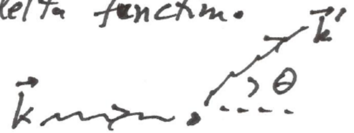
$E_k = \omega = |\vec{k}|$; $E_p = m$; $v_{rel} = c = 1$

$E_{k'} = \omega' = |\vec{k}'|$; $E_{p'} = \sqrt{m^2 + (\vec{p} + \vec{k} - \vec{k}')^2}$

$$= \sqrt{m^2 + (\vec{k} - \vec{k}')^2}$$

Integrate d^3p' using the delta function.

$d^3k' = k'^2 \, dk' \, d\Omega$



$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2} \frac{k'^2}{\omega m \omega' E'} \int dk' \, \delta(E' + k' - m - \omega)$$

$$\int dk'\, \delta[f(k')] = \frac{1}{|\partial f/\partial k'|}$$

$$f = E' + k' - m - \omega \quad ; \; E' = \sqrt{m^2 + k^2 + k'^2 - 2kk'\cos\theta}$$

$$\frac{\partial f}{\partial k'} = \frac{k' - k\cos\theta}{E'} + 1$$

$$= \frac{1}{E'}\left\{ k' - k\cos\theta + E' \right\}$$

$E' + k' = E + k = m + \omega$

$$= \frac{1}{E'}\left[m + \omega(1 - \cos\theta) \right]$$

<u>Conservation of energy</u> $\quad E' + \omega' = m + \omega$

$$E' = \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta}$$

$$m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta = m^2 + \omega^2 + \omega'^2 + 2m(\omega - \omega') - 2\omega\omega'$$

$$\omega'\left[m + \omega(1 - \cos\theta) \right] = m\omega$$

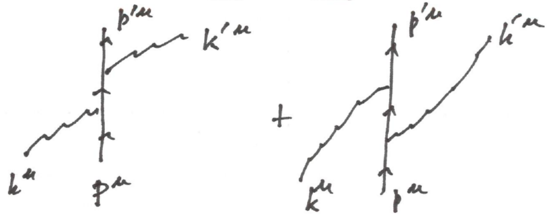
$$\omega' = \frac{m\omega}{\Delta} \quad \text{where} \quad \Delta = m + \omega(1 - \cos\theta)$$

$$\frac{\partial f}{\partial k'} = \frac{\Delta}{E'}$$

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2} \frac{\omega'}{\omega m E'} \frac{E'}{\Delta} = \frac{|M|^2}{64\pi^2} \left(\frac{\omega'}{m\omega} \right)^2$$

The transition matrix element

$\mathcal{M} =$



$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$$

$$\mathcal{M}_a = \bar{u}(p')\, e\gamma^{\mu}\, S_F(p+k)\, e\gamma^{\nu}\, u(p)$$

$$\epsilon^{*}_{\mu}(k')\ \epsilon_{\nu}(k)$$ and $\mathcal{M}_b$ similarly

where external photons have polarization vectors $\epsilon_{\mu}(k)$ (incoming) and $\epsilon^{*}_{\mu}(k')$ (outgoing)

See Peskin & Schroeder page

(We'll discuss quantization of the EM field next.)

For unpolarized Compton scattering we need

$$\frac{1}{2}\sum_{\text{pols.}} \frac{1}{2}\sum_{\text{spins.}} \left| \mathcal{M}_a + \mathcal{M}_b \right|^2 \equiv \overline{|\mathcal{M}|^2}$$

$$= \overline{|\mathcal{M}_a|^2} + \overline{|\mathcal{M}_b|^2} + \overline{\mathcal{M}_a \mathcal{M}_b^{*}} + \overline{\mathcal{M}_a^{*} \mathcal{M}_b}$$

The trick for summing over spin:

$$\sum_s u(ps)\,\bar{u}(ps) = \not{p} + m$$

The trick for summing over polarizations:

$$\sum_{\text{pol.}} \epsilon_\alpha(k)\,\epsilon^*_\beta(k) = g_{\alpha\beta} - \frac{k_\alpha k_\beta}{|\vec{k}|^2}$$

but $k_\alpha \tilde{j}^\alpha(k) = 0$ (charge is conserved);

So we just keep the $g_{\alpha\beta}$.

Example case: $\overline{|\mathcal{M}_a|^2}$

$\overline{|\mathcal{M}_a|^2}$ (including the pol. and spin. sums)

$$= \frac{e^4}{4}\,\mathrm{Tr}\; \gamma^\mu S_F(Q)\gamma^\nu(\not{p}+m)\gamma^\beta S_F(Q)\gamma^\alpha(\not{p}'+m)$$
$$g_{\mu\alpha}\, g_{\nu\beta}$$

Here $Q^\mu = p^\mu + k^\mu$; $Q^2 = m^2 + 2p\cdot k$

and $S_F(Q) = \dfrac{\not{Q}+m}{Q^2-m^2} = \dfrac{\not{Q}+m}{2p\cdot k}$

$$\overline{|\mathcal{M}_a|^2} = \frac{1}{4}\left(\frac{e^2}{2p\cdot k}\right)^2$$

$$\mathrm{Tr}\; \underbrace{\gamma^\mu(\not{Q}+m)\gamma^\nu(\not{p}+m)\gamma_\nu(\not{Q}+m)\gamma_\mu(\not{p}'+m)}_{\equiv Y}$$

Trace of 8 gamma matrices

But we can use some tricks. (Appendix)

$$\gamma^\mu \gamma_\mu = 4$$

$$\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$$

$$\gamma^\mu \not{p}_1 \not{p}_2 \gamma_\mu = 4\, p_1 \cdot p_2$$

Also, $\not{p}\not{p} = p^\alpha p^\beta \gamma_\alpha \gamma_\beta = p^\alpha p^\beta \{\gamma_\alpha, \gamma_\beta\}/2$

$$= p^2$$

$$Y = \gamma^\mu (\not{Q}+m) \underbrace{\gamma^\nu (\not{p}+m) \gamma_\nu}_{(-2\not{p}+4m)} (\not{Q}+m) \gamma_\mu$$

$$= 4m\, \gamma^\mu (\not{Q}+m)(\not{Q}+m) \gamma_\mu$$

$$\quad -2\, \gamma^\mu (\not{Q}+m) \not{p} (\not{Q}+m) \gamma_\mu$$

$$= 4m \left\{ 4(Q^2+m^2) + 2m(-2\not{Q}) \right\}$$

$$\quad -2\, \gamma^\mu \underbrace{(\not{Q}+m)\left[(-\not{Q}+m)\not{p} + 2p\cdot Q\right]}_{\left[-(Q^2-m^2)\not{p} + 2p\cdot Q\,(\not{Q}+m)\right]} \gamma_\mu$$

2nd line $\longrightarrow (-2)\left[(Q^2-m^2)\,2\not{p} + 2p\cdot Q\,(4m-2\not{Q})\right]$

$$\Upsilon = 16m(Q^2+m^2) - 16m^2\not{Q}$$
$$-2\left[(Q^2-m^2)\,2\not{p} + 2p\cdot Q\,(-2\not{Q}+4m)\right]$$

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$$\overline{|\mathcal{M}_a|^2} = \frac{1}{4}\left(\frac{e^2}{2p\cdot k}\right)^2 \mathrm{Tr}\ \Upsilon(\not{p}'+m)$$
$$= \frac{e^4}{16(p\cdot k)^2}\Big\{\big(16m(Q^2+m^2) - 16m\,p\cdot Q\big)\cdot 4m$$
$$+ (-16m^2 + 8p\cdot Q)\,4p'\cdot Q$$
$$- 4(Q^2-m^2)\,4p'\cdot p\Big\}$$

Lots of algebra to do

lab frame

$$Q^2 = (p+k)^2 = m^2 + 2p\cdot k \quad = m^2 + 2m\omega$$
$$p\cdot Q = p\cdot(p+k) = m^2 + p\cdot k \quad = m^2 + m\omega$$
$$p'\cdot Q = p'\cdot(p+k) = p\cdot p' + p\cdot k \quad = mE' + m\omega'$$
$$= m(m+\omega)$$
$$p'\cdot k = p\cdot k'$$

With the help of Mathematica ...

$$\overline{|\mathcal{M}_a|^2} = 32\, m^2 (m^2 + m\omega + \omega\omega') \times \frac{e^4}{16 m^2 \omega^2}$$

$$\overline{|\mathcal{M}_b|^2} = 32\, m^2 (m^2 - m\omega' + \omega\omega') \times \frac{e^4}{16 m^2 \omega'^2}$$

$$\overline{\mathcal{M}_a^* \mathcal{M}_b} = 16\, m^2 (2m^2 + m\omega - m\omega') \times \frac{e^4}{(-16 m^2 \omega\omega')}$$

$$\overline{\mathcal{M}_a \mathcal{M}_b^*} = \text{same}$$

$$\overline{|\mathcal{M}|^2} = e^4\, 2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 2m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + m^2 \left(\frac{1}{\omega} - \frac{1}{\omega'}\right)^2 \right]$$

Recall $\displaystyle \frac{1}{\omega} - \frac{1}{\omega'} = \frac{\cos\theta - 1}{m}$

$$\overline{|\mathcal{M}|^2} = e^4\, 2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta \right]$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2}\left(\frac{\omega'}{m\omega}\right)^2$$

$$= \frac{e^4}{32\pi^2 m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right]$$

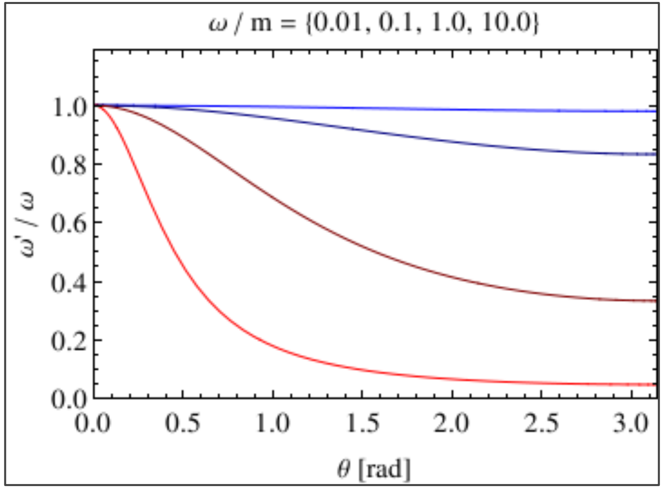
$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$$

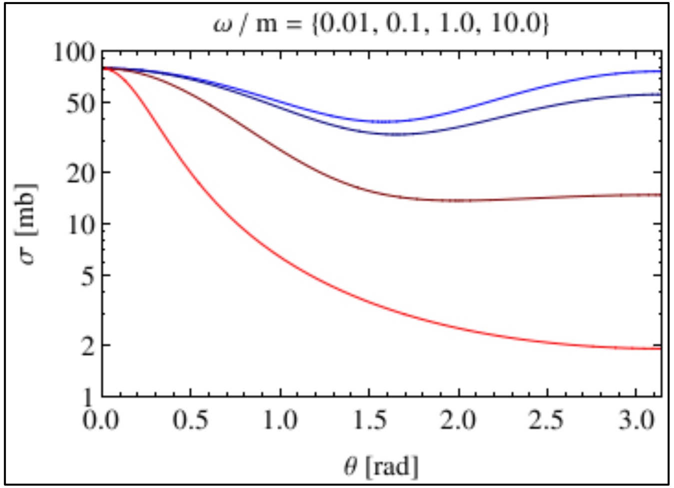
$$= \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right]$$

the Klein–Nishina formula

$$\frac{\alpha^2}{2(mc^2)^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right](\hbar c)^2$$

where $\omega' = \dfrac{m\omega}{m + \omega(1-\cos\theta)}$





The low energy limit is the Thomson cross section $(r_e^2/2) * (1+\cos^2\theta)$ where r_e = classical radius of the electron = 2.8 fm.