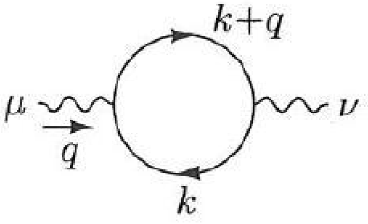
Overview of Charge Renormalization

Before beginning a detailed calculation, let's ask what kind of an answer we expect and what its interpretation will be. The interesting part of the diagram is the electron loop:

lowest order vacuum polarization



$$= (-ie)^2(-1)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^\mu \frac{i}{\not k - m}\gamma^\nu \frac{i}{\not k + \not q - m}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q). \tag{7.71}$$

(The fermion loop factor of (-1) was derived in Eq. (4.120).) More generally, let us define $i\Pi^{\mu\nu}(q)$ to be the sum of all 1-particle-irreducible insertions into the photon propagator,

all orders vacuum polarization

$$\equiv i\Pi^{\mu\nu}(q), \tag{7.72}$$

so that $\Pi_2^{\mu\nu}(q)$ is the second-order (in e) contribution to $\Pi^{\mu\nu}(q)$.

The only tensors that can appear in $\Pi^{\mu\nu}(q)$ are $g^{\mu\nu}$ and $q^\mu q^\nu$. The Ward identity, however, tells us that $q_\mu \Pi^{\mu\nu}(q) = 0$. This implies that $\Pi^{\mu\nu}(q)$ is proportional to the projector $(g^{\mu\nu} - q^\mu q^\nu/q^2)$. Furthermore, we expect that $\Pi^{\mu\nu}(q)$ will not have a pole at $q^2 = 0$; the only obvious source of such a pole would be a single-massless-particle intermediate state, which cannot occur in any 1PI diagram.[†] It is therefore convenient to extract the tensor structure from $\Pi^{\mu\nu}$ in the following way:

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2), \tag{7.73}$$

where $\Pi(q^2)$ is regular at $q^2 = 0$.

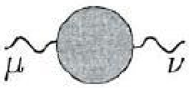
Using this notation, the exact photon two-point function is

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \cdots$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu \Pi(q^2) + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\sigma \Delta^\sigma_\nu \Pi^2(q^2) + \cdots,$$

where $\Delta^\rho_\nu \equiv \delta^\rho_\nu - q^\rho q_\nu / q^2$. Noting that $\Delta^\rho_\sigma \Delta^\sigma_\nu = \Delta^\rho_\nu$, we can simplify this expression to

sum the geometric series



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(\delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}\right)\left(\Pi(q^2) + \Pi^2(q^2) + \cdots\right)$$

$$= \frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right). \qquad (7.74)$$

In any S-matrix element calculation, at least one end of this exact propagator will connect to a fermion line. When we sum over all places along the line where it could connect, we must find, according to the Ward identity, that terms proportional to q_μ or q_ν vanish. For the purposes of computing S-matrix elements, therefore, we can abbreviate

$$= \frac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))}. \qquad (7.75)$$

modification of the photon propagator

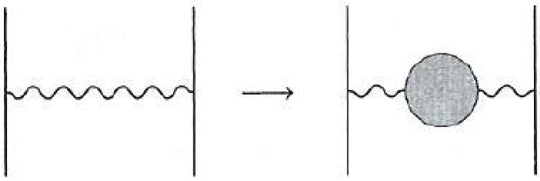
Notice that as long as $\Pi(q^2)$ is regular at $q^2 = 0$, the exact propagator always has a pole at $q^2 = 0$. In other words, the photon remains absolutely massless at all orders in perturbation theory. This claim depends strongly on our use of the Ward identity in (7.73). If, for example, $\Pi^{\mu\nu}(q)$ contained a term $M^2 g^{\mu\nu}$ (with no compensating $q^\mu q^\nu$ term), the photon mass would be shifted to M.

The residue of the $q^2 = 0$ pole is

$$\frac{1}{1-\Pi(0)} \equiv Z_3.$$

Z3 renormalization constant

The amplitude for any low-q^2 scattering process will be shifted by this factor, relative to the tree-level approximation:



$$\text{or} \qquad \cdots \frac{e^2 g_{\mu\nu}}{q^2} \cdots \longrightarrow \cdots \frac{Z_3\, e^2 g_{\mu\nu}}{q^2} \cdots.$$

Since a factor of e lies at each end of the photon propagator, we can conveniently account for this shift by making the replacement $e \to \sqrt{Z_3}\, e$. This replacement is called *charge renormalization*; it is in many ways analogous to the mass renormalization introduced in Section 7.1. Note in particular that the "physical" electron charge measured in experiments is $\sqrt{Z_3}\, e$. We will therefore shift our notation and call this quantity simply e. From now on we will refer to the "bare" charge (the quantity that multiplies $A_\mu \bar{\psi}\gamma^\mu\psi$ in the Lagrangian) as e_0. We then have

an example of charge renormalization

$$(\text{physical charge}) = e = \sqrt{Z_3}\, e_0 = \sqrt{Z_3} \cdot (\text{bare charge}). \tag{7.76}$$

To lowest order, $Z_3 = 1$ and $e = e_0$.

In addition to this constant shift in the strength of the electric charge, $\Pi(q^2)$ has another effect. Consider a scattering process with nonzero q^2, and suppose that we have computed $\Pi(q^2)$ to leading order in α: $\Pi(q^2) \approx \Pi_2(q^2)$. The amplitude for the process will then involve the quantity

$$\frac{-ig_{\mu\nu}}{q^2}\left(\frac{e_0^2}{1-\Pi(q^2)}\right) \underset{\mathcal{O}(\alpha)}{=} \frac{-ig_{\mu\nu}}{q^2}\left(\frac{e^2}{1-\left[\Pi_2(q^2)-\Pi_2(0)\right]}\right).$$

(Swapping e^2 for e_0^2 does not matter to lowest order.) The quantity in parentheses can be interpreted as a q^2-dependent electric charge. The full effect of replacing the tree-level photon propagator with the exact photon propagator is therefore to replace

$$\alpha_0 \to \alpha_{\text{eff}}(q^2) = \frac{e_0^2/4\pi}{1-\Pi(q^2)} \underset{\mathcal{O}(\alpha)}{=} \frac{\alpha}{1-\left[\Pi_2(q^2)-\Pi_2(0)\right]}. \tag{7.77}$$

(To leading order we could just as well bring the Π-terms into the numerator; but we will see in Chapter 12 that in this form, the expression is true to all orders when Π_2 is replaced by Π.)

Computation of Π_2

Having worked so hard to interpret $\Pi_2(q^2)$, we had better calculate it. Going back to (7.71), we have

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \text{tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right] \\ &= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}\big(k\cdot(k+q)-m^2\big)}{(k^2-m^2)\big((k+q)^2-m^2\big)}. \end{aligned} \tag{7.78}$$

We have written e and m instead of e_0 and m_0 for convenience, since the difference would give only an order-α^2 contribution to $\Pi^{\mu\nu}$.

Now introduce a Feynman parameter to combine the denominator factors:

$$\frac{1}{(k^2-m^2)\big((k+q)^2-m^2\big)} = \int_0^1 dx\, \frac{1}{(k^2+2xk\cdot q+xq^2-m^2)^2}$$

Performing a Wick rotation and substituting $\ell^0 = i\ell_E^0$, we obtain

$$i\Pi_2^{\mu\nu}(q) = -4ie^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \times \frac{-\frac{1}{2}g^{\mu\nu}\ell_E^2 + g^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left(m^2 + x(1-x)q^2\right)}{(\ell_E^2 + \Delta)^2}, \tag{7.79}$$

where $\Delta = m^2 - x(1-x)q^2$. This integral is very badly ultraviolet divergent. If we were to cut it off at $\ell_E = \Lambda$, we would find for the leading term,

$$i\Pi_2^{\mu\nu}(q) \propto e^2\Lambda^2 g^{\mu\nu},$$

with no compensating $q^\mu q^\nu$ term. This result violates the Ward identity; it would give the photon an infinite mass $M \propto e\Lambda$.

In the above analysis we regulated the divergent integral in the most straightforward and most naive way: by cutting it off at a large momentum Λ. But other regulators are possible, and some will in fact preserve the Ward identity. In our computations of the vertex and electron self-energy diagrams, we used a Pauli-Villars regulator. This regulator preserved the relation $Z_1 = Z_2$, a consequence of the Ward identity; a naive cutoff does not (see Problem 7.2). We could fix our present computation by introducing Pauli-Villars fermions. Unfortunately, several sets of fermions are required, making the method rather complicated.* We will use a simpler method, *dimensional regularization,* due to 't Hooft and Veltman.† Dimensional regularization preserves the symmetries of QED and also of a wide class of more general theories.

Next let us examine how $\widehat{\Pi}_2(q^2)$ modifies the electromagnetic interaction, as determined by Eq. (7.77). In the nonrelativistic limit it makes sense to compute the potential $V(r)$. For the interaction between unlike charges, we have, in analogy with Eq. (4.126),

$$V(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{-e^2}{|\mathbf{q}|^2[1-\widehat{\Pi}_2(-|\mathbf{q}|^2)]}. \tag{7.93}$$

Expanding $\widehat{\Pi}_2$ for $|q^2| \ll m^2$, we obtain

$$V(\mathbf{x}) = -\frac{\alpha}{r} - \frac{4\alpha^2}{15m^2}\delta^{(3)}(\mathbf{x}). \tag{7.94}$$

The correction term indicates that the electromagnetic force becomes much stronger at small distances. This effect can be measured in the hydrogen atom, where the energy levels are shifted by

$$\Delta E = \int d^3x\, |\psi(\mathbf{x})|^2 \cdot \left(-\frac{4\alpha^2}{15m^2}\delta^{(3)}(\mathbf{x})\right) = -\frac{4\alpha^2}{15m^2}|\psi(0)|^2.$$

The wavefunction $\psi(\mathbf{x})$ is nonzero at the origin only for s-wave states. For the $2S$ state, the shift is

$$\Delta E = -\frac{4\alpha^2}{15m^2}\cdot\frac{\alpha^3m^3}{8\pi} = -\frac{\alpha^5 m}{30\pi} = -1.123\times 10^{-7}\,\text{eV}.$$

This is a (small) part of the Lamb shift splitting listed in Table 6.1.

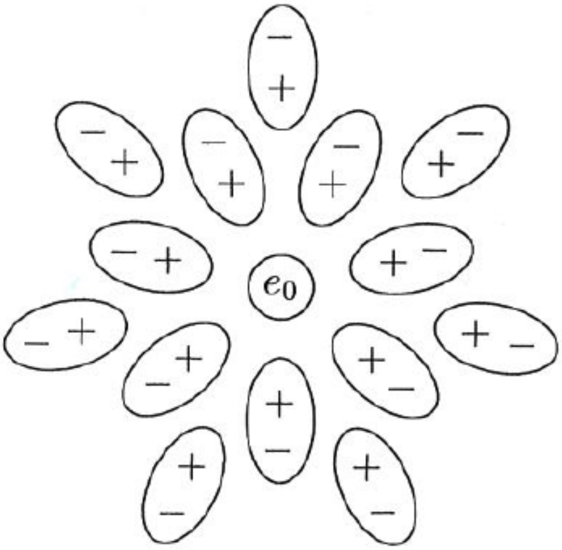


Figure 7.8. Virtual e^+e^- pairs are effectively dipoles of length $\sim 1/m$, which screen the bare charge of the electron.

so that

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{3/2}} + \cdots\right). \tag{7.95}$$

Thus the range of the correction term is roughly the electron Compton wavelength, $1/m$. Since hydrogen wavefunctions are nearly constant on this scale, the delta function in Eq. (7.94) was a good approximation. The radiative correction to $V(r)$ is called the *Uehling potential.*