# Chapter 9 : Functional Methods

We'll use this chapter to derive the equations for the quantized electromagnetic field.

## Sec. 9.1. Path integrals in Quantum Mechanics.

The path integral method is a different way to quantize classical mechanics -- an alternative to the Schroedinger equation.
Quick review

Ordinary quantum mechanics

$$H = \frac{p^2}{2m} + V(x) .$$

The operator method,

$$p = -i\hbar \frac{\partial}{\partial x} \quad \text{and} \quad -\frac{\hbar^2}{2m}\psi'' + V\psi = i\hbar \frac{\partial \psi}{\partial t}$$

etc.

The goal of quantum mechanics is to calculate the time evolution of states;

$$U(x_a, x_b ; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle$$

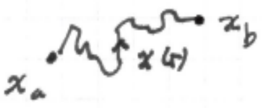
$$= \sum_n e^{-iE_nT/\hbar} \psi_n(x_b) \psi_n^*(x_a)$$

Feynman path integral formula

$$\langle x_b | e^{-iHT/\hbar} | x_a \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}$$

(eq. 9.3)

where $x(t)$ is a "path"




$x(0) = x_a$ and $x(T) = x_b$

and $S[x(t)]$ is the action

$$S[x(t)] = \int_0^T dt \left[ \frac{m}{2} \dot{x}^2 - V(x) \right]$$

Now, what is the "integral over paths"?

This is an example of functional integration.

To make it "rigorous", subdivide the time interval $(0,T)$ into discrete steps

$$(0, t_1, t_2, t_3, \ldots, t_{N-1}, T)$$

where $t_{k+1} - t_k = \epsilon$ ; i.e. discretize the path;

$$S = \sum_{k=1}^{N} \left[ \frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V\left( \frac{x_{k+1} + x_k}{2} \right) \right];$$

and $\mathcal{D}x(t)$ means $\prod_{k=1}^{N} dx_k$ ;

and take the limit $N \to \infty$ (i.e. $\epsilon \to 0$).

Evolution from state $|x_a\rangle$ to state $|x_b\rangle$ is equal to the integral over all paths from $x_a$ to $x_b$, weighted by the factor exp{i Action /hbar}.

Path integrals = the functional method applied to ordinary quantum mechanics.

*Now, **can** we apply functional integration to a field theory?*

## Sec. 9.2.
## Functional Quantization of Scalar Fields.

Let's compare and contrast: (i) the canonical quantization of the scalar field, and (ii) the functional formalism.

### (i) canonical quantization

$$\phi(x) = \phi(\vec{x}, t)$$

The action $S = \int L\, dt = \int \mathcal{L}\, d^4x$

where $\mathcal{L} = \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - V(\phi)$.

Lagrange's equation

$$\partial^2 \phi + \frac{\partial V}{\partial \phi} = 0$$

$$\frac{\partial^2}{\partial x^{0\,2}} - \nabla^2$$

Canonical momentum $\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi$.

Quantization $[\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y})$.

Time evolution; the generator is

$$H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right\}.$$

$$\therefore \langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = U[\phi_a, \phi_b; T]$$

### (ii) functional integration

In strict analogy with the path-integral formulation of ordinary quantum mechanics...

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int \underbrace{\mathcal{D}\phi(x)}_{\substack{\text{functional integral;}\\ \text{defined as the limit}\\ \epsilon \to 0 \text{ of a discretized field.}}} e^{\overbrace{i\int_0^T \mathcal{L}\, d^4x}}$$

action for a "path" in function space, from $\phi_a(\vec{x})$ to $\phi_b(\vec{x})$.

To see how this works, we'll consider the correlation function (2-point function)

$$D_F(x_1 - x_2) = \langle 0|\, T\, \varphi(x_1)\, \varphi(x_2)\, |0\rangle$$

in the ***free*** field theory.

This an ordinary function.
It should be the same in either formalism.

We already know what it is in the canonical field theory,

$$\mathcal{L}_{free} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow (\partial^2 + m^2)\phi = 0 \quad \text{(Klein Gordon equation)}$$

$$D_F(x_1 - x_2) = \langle 0|\, T\, \phi(x_1)\, \phi(x_2)\, |0\rangle$$
$$= \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x_1-x_2)}}{k^2 - m^2 + i\epsilon} \qquad \text{Eq (9.27).}$$

Now let's try to calculate D(x-x') using the functional method.

The functional integral for a correlation function is

$$\underbrace{\phi(x_1)\,\phi(x_2)} = \frac{\int \mathcal{D}\phi\ \phi(x_1)\,\phi(x_2)\, e^{iS(\phi)}}{\int \mathcal{D}\phi\ e^{iS(\phi)}}$$

(We want $S(\phi) = \int \mathcal{L}_{free}\, d^4x$ .)

Now,

$$S'(\phi) = \int d^4x \left\{ \tfrac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 \right\}$$

integrate by parts and throw away the surface term at $\infty$ $\Rightarrow$

$$S'(\phi) = \int d^4x \, \tfrac{-1}{2} \phi \left[ + \partial_\mu \partial^\mu + m^2 \right] \phi$$

$S(\phi)$ is quadratic (for the free field) so

$\int \mathcal{D}\phi \ e^{iS(\phi)}$ is a Gaussian integral.

Gaussian integration

Gaussian integration Consider $I^{(0)} =$

$$\int \left( \prod_{k=1}^{N} d\xi_k \right) e^{-\xi_i \cdot B_{ij} \cdot \xi_j} . \qquad \sum_{ij=1}^{N} \text{ implied}$$

$B_{ij}$ is symmetric ; eigenvalues

$$B_{ij} \, u_j^{(k)} = b_k \, u_i^{(k)} \qquad (k = 1 \dots N)$$

Expand $\xi_i = \sum_k c_k u_i^{(k)}$

$$\xi_i \cdot B_{ij} \cdot \xi_j = \sum_k b_k c_k^2$$

$$\boxed{\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}}$$

$$I^{(0)} = \int \left( \prod_k dc_k \right) e^{-\sum_k b_k c_k^2} = \prod_k \sqrt{\frac{\pi}{b_k}}$$

$$= \frac{\pi^{N/2}}{\sqrt{\operatorname{Det} B}} .$$

Also, consider $I^{(2)}_{rs} = \int \left(\prod_k d\xi_k\right) \xi_r \xi_s \, e^{-\xi \cdot B \cdot \xi}$

$$= \underbrace{\sum_{k'} c_{k'} \sum_{\ell'} c_{\ell'} \int \left(\prod_k dc_k\right) e^{-\sum_k b_k c_k^2}}_{\delta_{k'\ell'} \begin{cases} \int c_{k'}^2 e^{-b_{k'} c_{k'}^2} dc_{k'} = \frac{\sqrt{\pi}}{2 b_{k'}^{3/2}} \text{ for } k = k' \\ \int e^{-b_n c_n^2} dc_n = \sqrt{\frac{\pi}{b_n}} \text{ for } k = n \neq k' \end{cases}} \quad u_r^{(k')} u_s^{(\ell')}$$

$$\delta_{k'\ell'} \frac{1}{2 b_{k'}} \frac{\pi^{N/2}}{\sqrt{\operatorname{Det} B}}$$

$$= I^{(0)} \sum_{k'} \frac{u_r^{(k')} u_s^{(k')}}{2 b_{k'}}$$

$$= I^{0} \tfrac{1}{2} \left(B^{-1}\right)_{rs}$$

Apply Gaussian integration to the functional integral

$$\int \mathcal{D}\phi \, e^{iS(\phi)} = \int \mathcal{D}\phi \, e^{-\frac{i}{2}\int d^4x \, \phi(\partial^2+m^2)\phi}$$

$$= \text{const.} \times \left[\mathrm{Det}(\partial^2+m^2)\right]^{-1/2} \quad (\text{Eq } 9.25)$$

$$\int \mathcal{D}\phi \, \phi(x_1)\phi(x_2) \, e^{iS(\phi)} = \quad G(x_1,x_2) \int \mathcal{D}\phi \, e^{iS}$$

$\therefore$

$$\underbrace{\phi(x_1)\,\phi(x_2)} = G(x_1,x_2) = \text{inverse of } \partial^2+m^2$$

$$= \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x_1-x_2)} \frac{1}{k^2-m^2+i\epsilon}$$

Q.E.D.

**Results.**
(my proof was a little bit rough, but it can be made rigorous)

§ The two-point function from functional integration is the same as for canonical quantization.

§ The perturbation expansion can be carried out using the functional integrals (just expand $e^{iS}$ in powers of the interaction) and it's just the same as Wick's theorem.

§ The theorem can be extended to n-point functions.

***So… the functional method is a correct quantum theory.***

## Sec. 9.3.
## The analogy between quantum field theory and statistical mechanics.

In statistical mechanics, we sum over all the states of a large system. Think of the partition function Z = Sum $e^{-\beta E}$ .

In quantum field theory, we sum (i.e., integrate) over all the field configurations in space time:

$$\int \mathcal{D}\varphi\, e^{iS[\varphi]} .$$

## Sec. 9.4.
## Quantization of the electromagnetic field.

$A_\mu(x) = A_\mu(\mathbf{x},t)$

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu}F_{\mu\nu} - j^\mu A\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ ,

($j_\mu(x)$ is electric current density 4-vector, to acts as a source.
For free fields, $j_\mu = 0$.)

<u>Lagrange's equations are Maxwell's equations</u>

## Lagrange's equations are Maxwell's equations

$$\text{Action} = \int L\,dt = \int \mathcal{L}\, d^4x$$

$$= \int d^4x \left\{ -\tfrac{1}{4} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - j^\mu A_\mu \right\}$$

$$= \int d^4x \left\{ -\tfrac{1}{2} F^{\mu\nu} \partial_\mu A_\nu - j^\mu A_\mu \right\}$$

$$= \int d^4x \left\{ -\tfrac{1}{2} F^{0\nu} \partial_0 A_\nu - \tfrac{1}{2} F^{i\nu} \partial_i A_\nu - j^\mu A_\mu \right\}$$

integrate by parts; surface term at $\infty$ is zero

$$= \int d^4x \left\{ -\tfrac{1}{2} F^{0\nu} \partial_0 A_\nu + \tfrac{1}{2} (\partial_i F^{i\nu}) A_\nu - j^\mu A_\mu \right\}$$

$$\partial_0 \left( \frac{\delta L}{\delta(\partial_0 A_\rho)} \right) - \frac{\delta L}{\delta A_\rho} = 0$$

$$= \partial_0 \left( -\tfrac{1}{2} F^{0\rho} \times 2 \right) - \tfrac{1}{2} \partial_i F^{i\rho} \times 2 + j^\rho$$

$$= -\partial_\mu F^{\mu\rho} + j^\rho$$

Thus $\quad \partial_\mu F^{\mu\rho} = j^\rho \quad$ (Maxwell's equations)

## What is the problem with canonical quantization?

/1/ $A_\mu$ has 4 components.
But $\partial A_0/\partial t$ does not appear in the Lagrangian.
So the canonical momentum for $A_0$ is

$$\Pi = \delta L / \delta(\partial_0 A_0) = 0.$$

We can't apply canonical quantization to $A_0$.

/2/ The Maxwell equation corresponding to $A_0$ is

$$\partial_\mu F^{\mu 0} = j^0$$

$$\partial_i F^{i0} = j^0$$

$$\nabla \cdot \mathbf{E} = \rho \qquad \text{(Gauss's law, not a dynamical equation)}$$

/3/ One method is to impose the Coulomb gauge condition, $\nabla \cdot \mathbf{A} = 0$.
Then

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla A_0 - \partial \mathbf{A}/\partial t) = -\nabla^2 A_0 = \rho$$

so

$$A_0(x) = \int \frac{\rho(x')\, d^3x'}{4\pi |\vec{x} - \vec{x}'|} \quad \text{(Coulomb's law)}$$

I.e., the field $A_0$ is not quantized.
This method is used by Bjorken and Drell, and often in atomic physics. But it is not *manifestly* Lorentz invariant, so it is inconvenient for relativistic calculations.

/4/ Peskin and Schroeder use the functional method, and impose a covariant gauge condition on the integral over field configurations.

<u>The functional method</u>

For example, consider the correlation function

$$\int \mathcal{D}A_\rho \, A_\mu(x_1) A_\nu(x_2) e^{iS(A)} \Big/ \int \mathcal{D}A_\rho \, e^{iS(A)}$$

Now consider

$$\int \mathcal{D}A_\rho \, e^{iS(A)} \quad \text{where } S = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(free field)

But S is gauge invariant ;

$$S(A_\rho) = S(A_\rho + \partial_\rho \lambda)$$

for any function $\lambda(x)$.
The integral is *undefined* because the integrand is constant over an infinite space.

Another indication of the failure of the theory…

$$S(A) = \int \left(-\frac{1}{4}\right)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\, d^4x$$

$$= -\frac{1}{2}\int (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu)\, d^4x$$

integrate by parts; drop the surface term at ∞.

$$= -\frac{1}{2}\int A^\rho \left[\underbrace{-g_{\rho\sigma}\,\partial^2 + \partial_\rho\partial_\nu}_{\text{This operator is non invertible; the determinant is 0.}}\right] A^\sigma\, d^4x$$

**Challenge:** How to integrate over *gauge inequivalent configurations*, instead of integrating over all configurations.

Faddeev and Popov (1967)

$G(A)$ = gauge condition; e.g. $G = \partial_\mu A^\mu - \omega$

$$1 = \int \mathcal{D}\alpha\; \delta[G(A^\alpha)]\, \det\left[\frac{\delta G(A^\alpha)}{\delta \alpha}\right]$$

Analogy of the Dirac delta function; forces $G = 0$; i.e. $\partial_\mu A^\mu = \omega$

where $A_\mu^\alpha = A_\mu + \partial_\mu \alpha$

$$\int \mathcal{D}A\; e^{iS(A)} = \det\left[\frac{\delta G(A^\alpha)}{\delta\alpha}\right]\int D\alpha \int DA\, e^{iS(A)}\, \delta[G(A^\alpha)]$$

$$= \det\left[\frac{\delta G(A^\alpha)}{\delta\alpha}\right]\int D\alpha\int \mathcal{D}A'\; e^{iS(A')}\, \delta[G(A')]$$

change the variable of integration

$$= \det[\partial^2]\left(\int D\alpha\right)\int DA\, e^{iS(A)}\,\delta(\partial^\mu A_\mu - \omega)$$

↑ well defined because we're restricted the fields to satisfy the gauge condition

Valid for any $\omega(x)$.

Now integrate over $\omega(x)$ with a Gaussian weighting factor $\exp\left[-i\int d^4x\, \frac{\omega^2}{2\xi}\right] \longrightarrow$

$$= N(\xi)\det[\partial^2]\left(\int D\alpha\right)\int DA\, e^{iS(A)}\, e^{-i\int d^4x\, \frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$$\mathcal{L} \longrightarrow \mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \equiv \mathcal{L}_{eff}.$$

Now the effective action is

$$\int d^4x\, \mathcal{L}_{eff}(A) = -\frac{1}{2}\int A^\rho \left[-g_{\rho\sigma}\partial^2 + \partial_\rho\partial_\sigma\right] A^\sigma \, d^4x$$

$$- \frac{1}{2} \quad A^\rho \left[-\frac{1}{\xi}\partial_\rho\partial_\sigma \quad\right] A^\sigma \, d^4x$$

$$= -\frac{1}{2}\int A^\rho \underbrace{\left[-g_{\rho\sigma}\partial^2 + \left(1-\frac{1}{\xi}\right)\partial_\rho\partial_\sigma\right]}_{} A^\sigma \, d^3x$$

so the propagator is the inverse (Green's function) of this differential operator.

in momentum space,

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Check: (exercise)

$$\left[-g_{\mu\nu}k^2 + \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right]\tilde{D}_F^{\nu\rho}(k) = i\delta_\mu^{\ \rho}$$

We have used $\xi = 1$ "Feynman gauge"

Another common choice $\xi = 0$ "Landau gauge"

← same as Lorentz gauge

Gauge invariant calculations are always independent of $\xi$ ($\equiv$ gauge symmetry)

Ward identity $k_\mu \tilde{j}^\mu(k) = 0$ bec/ charge is conserved.

## Sec. 9.5.
## Functional Quantization of Spinor Fields.

Needs *Grassmann variables*, i.e., anticommuting numbers.

## Sec. 9.6.
## Symmetries in the Functional Formalism.