PHY 855 - Quantum Field Theory

Course description :

Introduction to field theory as it pertains to numerous problems in particle, nuclear and condensed matter physics. Second quantization, applications to different fields based on perturbation theory. Offered first half of semester.

Syllabus :

- theory of the photon
- Q.F.T. and many-particle systems
- prerequisites for relativistic Q.F.T. [phy 955]

condensed matter; nuclear physics

Textbooks :

- Mandl and Shaw, *Quantum Field Theory*
- Fetter and Walecka, *Quantum Theory of Many Particle Systems*
- Harris, *Pedagogical Approach to Q.F.T.*

Grading : 6 homework assignments

We'll start with something familiar.
The harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \quad \text{where} \quad [q, p] = i\hbar$$

Remember how we solve the harmonic oscillator.

We define a and a^+ by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(q + \frac{ip}{m\omega}\right) \text{ and } a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(q - \frac{ip}{m\omega}\right)$$

a = lowering operator = annihilation operator

a^+ = raising operator = creation operator

Note

$$q = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+) \text{ and } p = -i\sqrt{\frac{\hbar m\omega}{2}}\,(a - a^+)$$

Creation and annihilation of quanta

The crucial point is the commutator, $[a, a^{+}] = 1$.

That implies $[a^{+}a, a] = -a$

So a is the lowering operator

$$a|n\rangle \propto |n-1\rangle$$

and similarly a^{+} is the raising operator.

$$[a^{+}a, a] = a^{+}aa - aa^{+}a = [a^{+}, a]a = -a$$

($a^{+}a$ is the number operator)

Also, the Hamiltonian is

$$H = (a a^{+} + a^{+}a)\,\hbar\omega/2 = (a^{+}a + \tfrac{1}{2})\,\hbar\omega$$

Energy eigenstates

■ the ground state $|0>$

$$a|0> = 0 \;; \qquad H|0> = \tfrac{1}{2}\hbar\omega\,|0> \;; E_0 = \tfrac{1}{2}\,\hbar\omega$$

■ the first excited state $|1>$

$$|1> = a^+\,|0> \;; \qquad H|1>= 3/2\,\hbar\omega\,|0> \;; \qquad E_1 = 3/2\,\hbar\omega$$

■ all the eigenstates $|n>$ where $n \in \{0, 1, 2, 3, \dots\}$

$$|n> = (a^+)^n\,|0> /\mathrm{Sqrt}[n!]$$

$$E_n = (n + \tfrac{1}{2})\,\hbar\omega$$

$$\langle n|n\rangle = \frac{1}{n!}\langle 0|\,a^n (a^+)^n\,|0\rangle$$

$$= \frac{1}{n!}\langle 0|\,a^{n-1}\,\underbrace{a\,a^+}_{= 1 + a^+a = n}\,(a^+)^{n-1}\,|0\rangle$$

$$= \frac{1}{(n-1)!}\langle 0|\,a^{n-1}(a^+)^{n-1}\,|0\rangle$$

$$= \dots = \langle 0|0\rangle = 1.$$

The energy *eigenfunctions*

Recall the Dirac notation of bra's and ket's.

A state vector is a ket = $|\psi>$;

the wavefunction is $<x| \; \psi> = \psi(x)$.

So, the energy eigenfunctions are

$\Phi_n(x) = <x| n>$ $\qquad n \in \{ 0, 1, 2, 3, \ldots \}$

$\quad = C \times$ Hermite polynomial $\times$ gaussian function

Homework Problem 1. For the harmonic oscillator, derive the ground state wave function $\Phi_0(x)$ from this property of the ket, $a|0\rangle = 0$.

Time dependent states

Given $|\psi,0\rangle$ (= state at t=0) what is $|\psi,t\rangle$?

The Hamiltonian is the generator of translation in time; i.e.,

$$|\psi,t+\varepsilon\rangle = |\psi,t\rangle - i\,\varepsilon/\hbar\, H\, |\psi,t\rangle \quad ;$$

generator

or,

$$i\hbar\,(\partial/\partial t)\,|\psi,t\rangle = H\,|\psi,t\rangle \quad ;$$

Schroedinger equation

or,

$$|\psi,t\rangle = \exp(-\,i\,t\,H/\hbar\,)\,|\psi,0\rangle \quad ;$$

formal solution

or,

$$|\psi,t\rangle = \sum c_n \exp(-\,i\,t\,E_n/\hbar\,)\,|n\rangle \quad .$$

expand in |n>

What is a *classical* oscillator?

$$H = p^2/(2m) + \tfrac{1}{2} m \omega^2 x^2 \quad \Rightarrow \quad x''(t) + \omega^2 x(t) = 0.$$

$$x(t) = A \cos \omega t \qquad (\star)$$

How is that related to $|\psi,t\rangle$?

Of course $(\star)$ is impossible, *physically*, by the uncertainty principle. What I mean by $(\star)$ is that the uncertainty of x is small; i.e., small compared to A.

Comments.

- Define $x(t) = \langle \psi,t \mid x \mid \psi,t \rangle$
- By Ehrenfest's theorem, $x''(t) + \omega^2 x(t) = 0.$

But that's not good enough, because it doesn't say Δx is small !

The coherent state (R. Glauber)

Nobel Prize motivation (2005):
"for his contribution to the quantum theory of optical coherence"

For sure,

$$|\psi,t> = \sum c_n \exp(-i\, E_n\, t / \hbar\ )\ |n> .$$

The best description of a classical oscillator is

$$c_n = \exp(-\alpha^2/2)\ \alpha^n\ / \mathrm{Sqrt}(n!) \qquad .$$

Homework Problem 2. For the coherent state given in Lecture 1,
i.e., $c_n = \exp(-\alpha^2/2)\ \alpha^n\ / \mathrm{Sqrt}(n!)$:
(a) Calculate $\langle\, t \mid x \mid t\,\rangle = A \cos \omega t$. (Determine A.)
(b) Calculate $\langle\, t \mid x^2 \mid t\,\rangle$; and show that the uncertainty of x is small in the classical limit.
(c) Calculate $\langle\, t \mid H \mid t\,\rangle$. Compare the result to the classical energy.
Hint: |t> is an eigenstate of a.

Homework due Friday January 22

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