Fetter and Walecka

CHAPTER 3 : GREEN'S FUNCTIONS AND
FIELD THEORY (FERMIONS)

6. Pictures
7. Green's functions
8. Wick's theorem
9. Diagrammatic analysis of perturbation theory

Review

$$H = \int \psi^{\square}(x)\, T(x)\, \psi(x)\, d^3x$$

$$+\tfrac{1}{2} \iint \psi^{\square}(x)\, \psi^{\square}(x')$$

$$V(x,x')\ \psi(x')\, \psi(x)\, d^3x\, d^3x'$$

$$\{\psi(x)\,,\, \psi(x')\} = 0$$

$$\{\psi(x)\,,\, \psi^{\square}(x')\} = \delta^3(x-x')$$

(spin indices are suppressed)

6. PICTURES

The predictions of a quantum theory depend entirely on matrix elements;

$$< \alpha \mid Q \mid \beta > = Q_{\alpha\beta}(t)\ .$$

Now which parts of the theory (i.e., states or operators) depend on time?

Schroedinger picture: the states depend on time and the operators do not depend on time.

Heisenberg picture: the operators depend on time and the states do not depend on time.

Interaction picture: both states and operators depend on time.

The *matrix elements* , and hence *predictions*, must be equal in all three pictures. For example,

$$\langle \alpha_S(t) \mid Q_S \mid \beta_S\, (t) \rangle = \langle \alpha_H \mid Q_H\, (t) \mid \beta_H \rangle\ .$$

6a. The Schroedinger picture

This picture is the most familiar. The state depends on t, and is the solution of the time-dependent Schroedinger equation,

$$i\hbar\ \partial/\partial t\ |\ \Psi_S(t)\ \rangle = H\ |\ \Psi_S(t)\ \rangle$$

The formal solution of this equation is ...

$$|\Psi_s(t)\rangle = \underbrace{e^{-iH(t-t_0)/\hbar}}\ |\Psi_s(0)\rangle$$

H is Hermitian ($H^\dagger = H$)

so $e^{-iH(t-t_0)}$ is unitary ($U^\dagger U = 1$)

proof

Consider $\sigma_n = \left(-\frac{i(t-t_0)}{\hbar}\right)^n \frac{H^n}{n!}$

$$\frac{\partial \sigma_n}{\partial t} = \left(\frac{-i}{\hbar}\right)^n n(t-t_0)^{n-1}\frac{H^n}{n!}$$

$$= \left(-\frac{i}{\hbar}H\right)\left(-\frac{i(t-t_0)}{\hbar}\right)^{n-1}\frac{H^{n-1}}{(n-1)!}$$

$$= -\frac{i}{\hbar}H\ \sigma_{n-1}$$

Observables are time-independent Hermitian operators.
Matrix elements are
$O_{\alpha\beta}(t) = \langle\ \alpha, t\ |\ O\ |\ \beta, t\ \rangle$

6b. The Heisenberg picture

This picture is important for proving general theorems.

Consider this unitary transformation,

$$| \Psi_H > = e^{i H t / \hbar} | \Psi_S(t) > ;$$

and note that $| \Psi_H >$ does not depend on time t.
So, $| \Psi_H >$ does not obey the Schroedinger equation :

$$\frac{\partial}{\partial t} | \Psi_H > = e^{iHt/\hbar} \underbrace{\left(\frac{iH}{\hbar} - \frac{iH}{\hbar} \right)}_{= 0} | \Psi_S(t) >$$

The observables depend on time.

We must have

$$< \Psi_H | O_H(t) | \Psi_H > = < \Psi_S(t) | O_S | \Psi_S(t) >$$

$$= < \Psi_S(t) | \underbrace{e^{-iHt/\hbar} O_H(t) e^{iHt/\hbar}}_{\text{so this must be } = O_S} | \Psi_S(t) >$$

- $O_H(t) = e^{iHt/\hbar} O_S e^{-iHt/\hbar}$

Or,

$$\partial / \partial t \, O_H(t) = (i/\hbar) [H , O_H(t)]$$

Comment:
The Hamiltonian does not depend on time.

6c. The interaction picture

The interaction picture is useful for perturbation theory.

Assume $H = H_0 + H_I$,

where H_0 is *solvable* and H_I is a set of *interactions*, hopefully having small effects.

{For example, H_0 could be a single particle operator; and H_I could be a two-particle operator describing the interactions between particles.}

How can we calculate the effects of H_I?

Here is the definition of the interaction picture:

$$|\Psi_I(t)> = e^{iH_o t/\hbar} |\Psi_S(t)> ;$$

and

$$O_I(t) = e^{i H_o t/\hbar} O_S \, e^{-i H_o t/\hbar}$$

Homework Problem:
Show that matrix elements in the interaction and Schroedinger pictures are equal.

Solving for time evolution, using perturbation theory, in the interaction picture

$$|\Psi_I(t)\rangle = e^{iH_0t/\hbar}\,|\psi_S(t)\rangle$$ definition

$$= e^{iH_0t/\hbar}\, e^{-iH(t-t_0)/\hbar}\,|\psi_S(t_0)\rangle$$

evolution in the Schroedinger picture

$$= e^{iH_0t/\hbar}\, e^{-iH(t-t_0)/\hbar}\, e^{-iH_0t/\hbar}\,|\psi_I(t_0)\rangle$$

definition

$$= \hat{U}(t,t_0)\,|\Psi_I(t_0)\rangle$$

where

$$\hat{U}(t,t_0) = e^{iH_0t/\hbar}\, e^{iH(t-t_0)/\hbar}\, e^{-iH_0t/\hbar}$$

/Important: $e^A e^B \neq e^{A+B}$ for operators./

A differential equation for $\hat{U}$,

$$\frac{\partial \hat{U}}{\partial t} = e^{iH_0t/\hbar}\left(\frac{iH_0}{\hbar} - \frac{iH}{\hbar}\right) e^{-iH(t-t_0)/\hbar}\, e^{-iH_0t_0/\hbar}$$

$$= -\frac{i}{\hbar}\, H_I(t)\; e^{iH_0t/\hbar}\, e^{-iH(t-t_0)/\hbar}\, e^{-iH_0t_0/\hbar}$$

$\hookrightarrow$ The interaction Hamiltonian, in the interaction picture; $e^{iH_0t}\, H_I\, e^{-iH_0t}$ $(\hbar = 1)$

$$= -\frac{i}{\hbar}\, H_I(t)\, U(t,t_0)$$

Solve by iteraction $\Rightarrow$ perturbation theory

Solution by iteration …

$$\frac{\partial \hat{U}}{\partial t} = \frac{-i}{\hbar} H_I(t) U(t, t_0)$$

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_I(t')\hat{U}(t', t_0)\, dt'$$

That satisfies the diff eq., by the fundamental Theorem of calculus.
Also, $U(t_0, t_0) = 1$.

Iteration

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_I(t')\left\{1 - \frac{i}{\hbar}\int_{t_0}^{t'} H_I(t'')\hat{U}(t'', t_0)\, dt''\right\} dt'$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_I(t')\, dt'$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t'} H_I(t') H_I(t'') \underbrace{\hat{U}(t'', t_0)}_{\text{Iterate again}} dt''\, dt'$$

$$1 - \frac{i}{\hbar}\int_{t_0}^{t''} H_I(t''')\hat{U}(t''', t_0)\, dt'''$$

Result – a series of increasing powers of $H_I(t)$

$$\hat{U}(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_I(t')\, dt' \qquad 0^{\text{th}} + 1^{\text{st}} \text{ order}$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t'} H_I(t') H_I(t'')\, dt'\, dt'' \qquad 2^{\text{nd}} \text{ order}$$

$$+ \left(\frac{-i}{\hbar}\right)^3 \int_{t_0}^{t}\int_{t_0}^{t'}\int_{t_0}^{t''} H_I(t') H_I(t'') H_I(t''')\, dt'\, dt''\, dt''' \qquad 3^{\text{rd}} \text{ order}$$

$$+ \cdots \qquad \vdots$$

Or,

$$\hat{U}(t, t_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \cdots \int_{t_0}^{t^{(n-1)}} dt^{(n)}$$

$$H_I(t')\, H_I(t'')\, H_I(t''') \cdots H_I(t^{(n)})$$

Or,

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' \cdots \int_{t_0}^{t^{(n-1)}} dt^{(n)}$$

$$H_I(t')\, H_I(t'')\, H_I(t''') \cdots H_I(t^{(n)})$$

Time Ordering

The H_I's are time ordered: earlier times stand to the right of later times

$$t' > t'' > t''' > \cdots > t^{(n)}$$

Define the TIME ORDERED PRODUCT

$$T\left[H_I(t_1)\, H_I(t_2)\, H_I(t_3) \cdots H_I(t_n) \right]$$

$$\equiv H_I(t_1')\, H_I(t_2')\, H_I(t_3') \cdots H_I(t_n')$$

where

$\{t_1'\ t_2'\ t_3' \ldots t_n'\}$ = the permutation of $\{t_1\ t_2\ t_3 \ldots t_n\}$ such that the (t')'s are ordered in time.

$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_3 \cdots \int_{t_0}^{t} dt_n$$

$$T\left[H_I(t_1)\, H_I(t_2)\, H_I(t_3) \cdots H_I(t_n) \right] \times 1/n!$$

That's the perturbation expansion.

Or, formally

$$\hat{U}(t,t_0) = T\, e^{-\frac{i}{\hbar}\int_{t_0}^{t} H_I(t')\,dt'}$$

6d. Adiabatic "switching on"

Write $H = H_0 + H_I\, e^{-\varepsilon|t|}$;

and let $\varepsilon \to 0$ at the end of the calculations.

Acceptable results must have valid limits as $\varepsilon \to 0$.

The initial and final states ,
i.e., as $t \to -\infty$ and $+\infty$, respectively,
are eigenstates of H_0 ;
i.e., non-interacting particles.

The state experiences the interactions H_I during the time interval $-1/\varepsilon \lesssim t \lesssim +1/\varepsilon$.

6e. A theorem of Gell-Mann & Low

This is a bit of a technicality.

It states that the limiting process $\varepsilon \to 0$ is OK , despite potential divergences.

The state defined by the ratio

$$| \Psi (t=0) \rangle_\varepsilon \;/\; \langle \phi_0 | \Psi (t=0) \rangle_\varepsilon$$

is well defined as $\varepsilon \to 0$;
and it is an eigenstate of the full Hamiltonian, H.

(ϕ_0 means the free particle state at $t = -\infty$.)

Homework due Friday February 12

Problem 18.

Derive this equation for time evolution in the Interaction Picture:

$$i \hbar \, \partial / \partial t \; | \Psi_I(t) > = \textbf{\textit{(what?)}}$$

Problem 19.

Prove that $< a,t \,|\, O(t) \,|\, b,t >$

is the same in the interaction picture and in the Schroedinger picture.