Chapter 3 : GREEN'S FUNCTIONS AND FIELD THEORY (FERMIONS)

Review

We've developed a nice formal theory for describing many-particle systems. But what can we actually calculate?

What would we *want* to calculate? (Quantities that can be compared to experimental measurements !)

For realistic theories, we can't calculate anything exactly ---approximations are necessary.

8. WICK'S THEOREM

Wick's theorem is a formal result that will simplify calculations in perturbation theory.

<u>Statement of the theorem</u>
Any time-ordered product of operators can be expressed as the sum of normal-ordered products multiplied by c-number contractions.

$$
\begin{aligned}
T[\,A\,B\,C\,\ldots\,Z\,] &= N[\,A\,B\,C\,\ldots\,Z\,] \\
&\quad + x_{AB}\, N[\,C\,D\,E\,\ldots\,Z\,] + \text{similar terms} \\
&\quad + x_{AB}\, x_{CD}\; N[\,E\,F\,G\,\ldots\,Z\,] + \text{similar terms} \\
&\quad + \text{all the rest}
\end{aligned}
$$

Why is that useful?
Because the ground-state expectation value of any normal-ordered product is 0.

Preliminaries

(to motivate the importance of Wick's theorem)

The Green's function is the ground state expectation value of a certain operator.

(The operator is the time ordered product of two field operators.)

First consider a general problem

Start with the Heisenberg picture,

$$\langle 0 | O_H(t) | 0 \rangle.$$

Now write this in the interaction picture

(so that we can apply perturbation theory).

$$\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle / \langle \Psi_0 | \Psi_0 \rangle$$

By the Gell Mann & Low theorem

$$\frac{|\Psi_0\rangle}{\langle \Phi_0 | \Psi_0 \rangle} = \frac{U_\epsilon(t=0; \pm\infty) | \Phi_0 \rangle}{\langle \Phi_0 | U_\epsilon(t=0; \pm\infty) | \Phi_0 \rangle}$$

(and we take the limit $\epsilon \to 0$)

So

$$\frac{\langle \Psi_0 | \hat{O}_H^{(t)} | \Psi_0 \rangle}{|\langle \Phi_0 | \Psi_0 \rangle|^2} = \frac{\langle \Phi_0 | U_\epsilon^\dagger(\infty, 0) \hat{O}_H^{(t)} U_\epsilon(0, -\infty) | \Phi_0 \rangle}{\langle \Phi_0 | U_\epsilon(0, \infty) | \Phi_0 \rangle^* \langle \Phi_0 | U_\epsilon(0, -\infty) | \Phi_0 \rangle}$$

$$\hat{O}_H(t) = \underbrace{U_\epsilon(0, t)}_{U_\epsilon^\dagger(t,0)} \hat{O}_I(t) U_\epsilon(t, 0)$$

$$\hookrightarrow = \langle \Phi_0 | U_\epsilon(\infty, t) \hat{O}_I(t) U_\epsilon(t, -\infty) | \Phi_0 \rangle / \text{den.}$$

Recall,

$U_\varepsilon(t_b,t_a) = T \exp \left[-i/\hbar \int_{ta}^{tb} H_I(t') \, e^{-\varepsilon|t'|} \, dt' \right]$

$$\frac{\langle \Psi_0 | \hat{O}_H(t) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \langle \Phi_0 | U_\epsilon(\infty, t) \, \hat{O}_I(t) \, U_\epsilon(t, -\infty) | \Phi_0 \rangle / \text{den.}$$

↑ Heisenberg picture

↑ everything here is in the interaction picture

Recall $U_\epsilon(t_2, t_1) = T \exp\left[\frac{-i}{\hbar} \int_{t_1}^{t_2} H_I(t') e^{-\epsilon|t'|} dt' \right]$

Therefore we'll have these operators

$$\underbrace{T[H_1 H_1 H_1 \dots H_1]}_{\leftarrow n \text{ terms} \rightarrow} \hat{O}_I(t) \, \underbrace{T[H_1 H_1 \dots H_1]}_{\leftarrow m \text{ terms} \rightarrow}$$

which we can replace by

$$T[\underbrace{H_1 H_1 \dots H_1 H_1}_{\leftarrow n+m \text{ terms} \rightarrow} \hat{O}_I(t)]$$

Hence,

$\langle \Psi_0 | O_H(t) | \Psi_0 \rangle \, / \, \langle \Psi_0 | \Psi_0 \rangle$

$= \sum_{\nu=0}^{\infty} (-i/\hbar)^\nu \; 1/\nu! \quad \int dt_1 \, e^{-\varepsilon|t_1|} \int dt_2 \, e^{-\varepsilon|t_2|} \dots \int dt_\nu \, e^{-\varepsilon|t_\nu|}$

$\langle \Phi_0 | T[H_I(t_1) H_I(t_2) \dots H_I(t_\nu) \;\; O_I(t)] | \Phi_0 \rangle$

$/ \; \langle \Phi_0 | S | \Phi_0 \rangle$

where $\int dt_i$ means $\int_{-\infty}^{\infty}$.

Similarly for the 1-particle GREEN's FUNCTION

$$i G_{\alpha\beta}(x,y) = \frac{\langle \Phi_0 | T[\psi_\alpha(x)\psi_\beta^\dagger(y)] | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle}$$

$x = (\vec{x}, t_x)$ and $y = (\vec{y}, t_y)$

$$= \sum_{\nu=0}^{\infty} \left(\frac{-i}{\hbar}\right)^\nu \frac{1}{\nu!} \int dt_1\, dt_2 \cdots dt_\nu$$

$$\langle \Phi_0 | T[H_1(t_1) H_1(t_2) \ldots H_1(t_\nu)\, \psi_\alpha(x)\, \psi_\beta^\dagger(y)] | \Phi_0 \rangle \Big/ \text{den.}$$

$\Rightarrow$ perturbation expansion

$$i G_{\alpha\beta}(x,y) = \Big\{ \; i G^0_{\alpha\beta}(x y)$$

$$+ \left(\frac{-i}{\hbar}\right) \int_{-\infty}^{\infty} dt_1 \, \langle \Phi_0 | T[H_1(t_1)\psi_\alpha(x)\psi_\beta^\dagger(y)] | \Phi_0 \rangle$$

$$+ \cdots\cdots \Big\} \Big/ \text{den.}$$

all the operators on the R.H.S. are interaction picture operators.

So, the problem reduces to calculation of

$$\langle \Phi_0 | T[\psi\psi \ldots \psi \; \psi^\dagger \psi^\dagger \ldots \psi^\dagger] | \Phi_0 \rangle$$

in the interaction picture

Time ordering and normal ordering

We already know time ordering...

★ $T(A' B' C' D' \ldots) = (-1)^P \times (A\,B\,C\,D \ldots)$
where $\{A\,B\,C\,D \ldots\}$ are in time order.

Now, what is normal ordering?

Assume (as is usually the case) that the field operator has both annihilation terms and creation terms.

- Example: In relativistic QED, $\psi(x)$ annihilates electrons and creates positrons.
- Example: In the quantum theory of metals, $\psi(x)$ annihilates electrons above the Fermi energy ("particles") and creates "holes" below the Fermi energy.
- Example: In the the nuclear shell model, $\psi(x)$ annihilates nucleons above the filled shells ("particles") and creates holes in the filled shells.

So, we can write

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$$
= annihilation part plus creation part;
note $\psi^{(+)}(x)|\Phi_0\rangle = 0$.

Also,

$$\psi^\dagger(x) = \psi^{(+)\dagger}(x) + \psi^{(-)\dagger}(x)$$
= creation part plus annihilation part;
note $\psi^{(-)\dagger}(x)|\Phi_0\rangle = 0$.

★ A product of field operators is in normal order if all the annihilation operators stand to the right of all creation operators.

★ $N(A' B' C' \ldots) = (-1)^P \times (A\,B\,C \ldots)$ where $\{A\,B\,C \ldots\}$ are in normal order.

Theorem. The expectation value in Φ_0, of a normal ordered product, is 0.

Wick's theorem

T(U V W … X Y Z)

= N(U V W … X Y Z) + all possible pairs of contractions.

See any quantum field theory textbook for the general proof.

Proof by examples (assuming fermions)

Suppose U V W are annihilation parts at later times than X Y Z which are all creation parts.

Consider

$$\Xi = T(U V W \ X Y Z) = U V W \ X Y Z.$$

But this is not in normal order.

Move X to the left using the commutation relations.

$\Xi = U V W \ X Y Z = UV (\{W,X\} - XW) YZ$

$= - UVXWYZ + c(W,X) \ UVYZ$

(the contraction is a c number)

In the first term move X to the left; in the second term move Y to the left.

$\Xi = - (- UXVWYZ \ + c(V,X) \ UWYZ)$
$\quad + c(W,X) (- UYVZ + c(V,Y) \ UZ)$

keep going, always moving creation parts to the left

$\Xi = - XUVWYZ + c(U,X) \ VWYZ$
$\quad -c(V,X) (-UYWZ + c(W,Y) \ UZ)$
$\quad -c(W,X) (-YUVZ + c(Y,V) \ UZ)$
$\quad +c(W,X)c(V,Y) (- ZU + c(U,Z))$

until all the terms are in normal order.

$\Xi = - XYZUVW + c(U,X) \ YZVW$ + many similar
$\quad -c(U,X) \ c(W,Y) \ ZV$ + many similar
$\quad +c(W,X) \ c(V,Y) \ c(U,Z)$ + many similar

= N(UVWXYZ) + all possible pairs of contractions.

Now, what are the contractions?

In the interaction picture,

$$\hat{\psi}(\vec{x}t) = \sum_{\vec{k}\lambda} \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{\Omega}} u_\lambda e^{-i\omega_k t} \left[\underbrace{\theta(k-k_F) a_{\vec{k}\lambda}}_{\psi^{(+)}} + \underbrace{\theta(k_F-k) b^\dagger_{-\vec{k}\lambda}}_{\psi^{(-)}} \right]$$

$$\hat{\psi}^\dagger(\vec{x},t) = \sum_{\vec{k}\lambda} \frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{\Omega}} u^\dagger_\lambda e^{i\omega_k t} \left[\underbrace{\theta(k-k_F) a^\dagger_{\vec{k}\lambda}}_{\psi^{(+)\dagger}} + \underbrace{\theta(k_F-k) b_{-\vec{k}\lambda}}_{\psi^{(-)\dagger}} \right]$$

$\psi^{(\pm)} \begin{cases} \text{annihilation} \\ \text{creation} \end{cases}$ $\quad \psi^{(\pm)\dagger} \begin{cases} \text{creation} \\ \text{annihilation} \end{cases}$

- $\{ \psi^{(\pm)}(x), \psi^{(\pm)}(y) \} = 0$ because $\{a, a\} = 0$, $\{a, b^\dagger\} = 0$, $\{b^\dagger, a\} = 0$, $\{b^\dagger, b^\dagger\} = 0$
- $\{ \psi^{(\pm)\dagger}(x), \psi^{(\pm)\dagger}(y) \} = 0$ similarly
- $\{ \psi^{(+)}(x), \psi^{(-)\dagger}(y) \} = 0$ because $\{a, b\} = 0$
- $\{ \psi^{(-)}(x), \psi^{(+)\dagger}(y) \} = 0$ because $\{b^\dagger, a^\dagger\} = 0$

$\therefore$ There are only 2 nonzero contractions.

$$c\left(\psi^{(+)}_\alpha(x), \psi^{(+)\dagger}_\beta(y) \right)$$

$$= T\left[\psi^{(+)}_\alpha(x)\, \psi^{(+)\dagger}_\beta(y) \right] - N\left[\psi^{(+)}_\alpha(x)\, \psi^{(+)\dagger}_\beta(y) \right]$$

If $t_x > t_y$ then

$$c = \psi^{(+)}_\alpha(x)\, \psi^{(+)\dagger}_\beta(y) + \psi^{(+)\dagger}_\beta(y)\, \psi^{(+)}_\alpha(x)$$

$$= i G^0_{\alpha\beta}(x,y) \text{ for } t_x > t_y \quad \text{(Homework)}$$

If $t_x < t_y$ then

$$c = -\psi^{(+)\dagger}_\beta(y)\, \psi^{(+)}_\alpha(x) + \psi^{(+)\dagger}_\beta(y)\, \psi^{(+)}_\alpha(x)$$

$$= 0$$

Result

$$c\left(\psi^{(+)}_\alpha(x), \psi^{(+)\dagger}_\beta(y) \right) = \begin{cases} i G^0_{\alpha\beta}(xy) & \text{for } t_x > t_y \\ 0 & \text{for } t_x < t_y \end{cases}$$

and similarly

$$c\left(\psi^{(-)}_\alpha(x), \psi^{(-)\dagger}_\beta(y) \right) = \begin{cases} 0 & \text{for } t_x > t_y \\ i G^0_{\alpha\beta}(xy) & \text{for } t_x < t_y \end{cases}$$

THEREFORE

$$c\left(\psi_\alpha(x), \psi^\dagger_\beta(y) \right) = i G^0_{\alpha\beta}(xy) \quad \text{Eq. (8.29)}$$

The Wick contraction is equal to the propagator function.

Feynman diagrams

Calculate the Green's function in perturbation theory.

$iG_{\alpha\beta}(x,y) = \langle\Phi_0|\, T\,[\psi_\alpha(x)\, \psi_\beta^\dagger(y)]\, |\Phi_0\rangle$ (H. picture)

x means $(t_x,\mathbf{x})$ and y means $(t_y,\mathbf{y})$.

$= \sum_{\nu=0}^{\infty} (-i/\hbar)^\nu\ 1/\nu!\ \int dt_1\, dt_2 \ldots dt_\nu$

$\langle 0|T\,[\,H_I(t_1)\, H_I(t_2) \ldots H_I(t_\nu)\, \psi_\alpha(x)\, \psi_\beta^\dagger(y)]\, |0\rangle$

(Int. picture)

$= \sum_{\nu=0}^{\infty} (-i/\hbar)^\nu\ 1/\nu!\ \int dt_1\, dt_2 \ldots dt_\nu$

{ *the sum of all complete contractions* }

i.e., all Feynman diagrams

Homework Problem due Friday, Feb 19

Problem 21.

The Wick contraction of $\psi_\alpha(x)$ and $\psi_\beta^\dagger(y)$ is, by Wick's theorem,

$c(\,\psi_\alpha(x)\,,\, \psi_\beta^\dagger(y)\,)$

$= T[\,\psi_\alpha(x)\, \psi_\beta^\dagger(y)\,] - N[\,\psi_\alpha(x)\psi_\beta^\dagger(y)\,]$

Prove:

$c(\,\psi_\alpha(x)\,,\, \psi_\beta^\dagger(y)\,) = i\, G_{\alpha\beta}^0(x,y)$