Chapter 3 : GREEN’S FUNCTIONS AND FIELD THEORY (FERMIONS)

Review

We've developed a nice formal theory for describing many-particle systems. But what can we actually calculate?

What would we want to calculate? (Quantities that can be compared to experimental measurements!)

For realistic theories, we can't calculate anything exactly ---approximations are necessary.

8. WICK’S THEOREM

Wick's theorem is a formal result that will simplify calculations in perturbation theory.

Statement of the theorem
Any time-ordered product of operators can be expressed as the sum of normal-ordered products multiplied by c-number contractions.

\[ T[A_1 A_2 A_3 \ldots Z] = N[A_1 A_2 A_3 \ldots Z] + x_{AB} N[C_1 C_2 \ldots Z] + \text{similar terms} \]

\[ + x_{AB} x_{CD} N[E_1 E_2 \ldots Z] + \text{similar terms} \]

\[ + \text{all the rest} \]

Why is that useful?
Because the ground-state expectation value of any normal-ordered product is 0.
Preliminaries
(to motivate the importance of Wick’s theorem)

The Green’s function is the ground state expectation value of a certain operator. (The operator is the time ordered product of two field operators.)

First consider a general problem
Start with the Heisenberg picture,
\[ \langle 0 | O_H(t) | 0 \rangle. \]
Now write this in the interaction picture (so that we can apply perturbation theory).
Hence,

\[
\langle \Psi_0 | O_H(t) | \Psi_0 \rangle / \langle \Psi_0 | \Psi_0 \rangle = \sum_{\nu=0}^{\infty} \left( -\frac{i}{\hbar} \right)^{\nu/\nu!} \int dt_1 e^{-\varepsilon|t_1|} \int dt_2 e^{-\varepsilon|t_2|} \ldots \int dt_{\nu} e^{-\varepsilon|t_{\nu}|}
\]

where \( \int dt_i \) means \( \int_{-\infty}^{\infty} \).
Similarly for the 1-particle Green's function

\[ \mathbf{i} G_{\lambda \lambda}(x,y) = \frac{\langle \Phi_0 | T [ \psi_\lambda(x) \psi^\dagger_\lambda(y) ] | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle} \]

\[ x = (x, t_x) \text{ and } y = (y, t_y) \]

\[ = \sum_{\nu=0}^{\infty} \left( \frac{-\hbar}{i} \right)^{\nu+1} \int dt_1 \int dt_2 \cdots \int dt_{\nu} \]

\[ \langle \Phi_0 | T [ \psi_\lambda(t_1) \psi^\dagger_\lambda(t_2) \cdots \psi_\lambda(t_\nu) \psi^\dagger_\lambda(t_{\nu+1}) | \Phi_0 \rangle \]
Time ordering and normal ordering

We already know time ordering...

\[ T(A' B' C' D' \ldots) = (-1)^P \times (A \ B \ C \ D \ldots) \]
where \{A \ B \ C \ D \ldots\} are in time order.

Now, what is normal ordering?

Assume (as is usually the case) that the
field operator has both annihilation terms
and creation terms.

\[ \text{Example: In relativistic QED, } \psi(x) \text{ annihilates electrons and creates positrons.} \]
\[ \text{Example: In the quantum theory of metals, } \psi(x) \text{ annihilates electrons above the Fermi energy} \]
\[ \text{ (“particles”) and creates “holes” below the Fermi energy.} \]
\[ \text{Example: In the the nuclear shell model, } \psi(x) \text{ annihilates nucleons above the filled shells} \]
\[ \text{ (“particles”) and creates holes in the filled shells.} \]

So, we can write

\[ \psi(x) = \psi^+(x) + \psi^-(x) \]
\[ = \text{annihilation part plus creation part;} \]
\[ \text{note } \psi^+(x)|\Phi_0> = 0. \]

Also,

\[ \psi^\dagger(x) = \psi^+\dagger(x) + \psi^-\dagger(x) \]
\[ = \text{creation part plus annihilation part;} \]
\[ \text{note } \psi^-\dagger(x)|\Phi_0> = 0. \]

★ A product of field operators is in
normal order if all the annihilation
operators stand to the right of all
creation operators.

\[ N(A' B' C' \ldots) = (-1)^P \times (A \ B \ C \ldots) \]
where \{A \ B \ C \ldots\} are in normal order.

Theorem. The expectation value in \(\Phi_0\), of a
normal ordered product, is 0.
Wick’s theorem

\[ T(U V W \ldots X Y Z) = N(U V W \ldots X Y Z) + \text{all possible pairs of contractions.} \]

See any quantum field theory textbook for the general proof.

Proof by examples (assuming fermions)

Suppose \( U V W \) are annihilation parts at later times than \( X Y Z \) which are all creation parts.

Consider

\[ \Xi = T(U V W X Y Z) = U V W X Y Z. \]

But this is not in normal order.

Move \( X \) to the left using the commutation relations.

\[ \Xi = UVW X Y Z = UV (\{W,X\} - XW) YZ \]

\[ \Xi = -UVXWYZ + c(W,X) UVYZ \]

(the contraction is a \( c \) number)

In the first term move \( X \) to the left; in the second term move \( Y \) to the left.

\[ \Xi = -(-UXVWYZ + c(V,X) UWYZ) + c(W,X) (-UYVZ + c(V,Y) UZ) \]

keep going, always moving creation parts to the left

\[ \Xi = -XUVWYZ + c(U,X) VWYZ \]

\[ -c(V,X) (-UYWZ + c(W,Y) UZ) \]

\[ -c(W,X) (-UYVZ + c(Y,V) UZ) \]

\[ +c(W,X)c(V,Y) (-ZU + c(U,Z)) \]

until all the terms are in normal order.

\[ \Xi = -XYZUVW + c(U,X) YZVW + \text{many similar} \]

\[ -c(U,X) c(W,Y) ZV + \text{many similar} \]

\[ +c(W,X) c(V,Y) c(U,Z) + \text{many similar} \]

\[ = N( U VWXYZ) + \text{all possible pairs of contractions.} \]

Q.E.D.
Now, what are the contractions?

In the interaction picture,
\[ \hat{\Psi}^{(\pm)}(x) = \sum_{\alpha, \beta} \frac{e^{i \frac{\mathbf{p}_\alpha \cdot \mathbf{x}}{\hbar}}} {\sqrt{E_\alpha \sqrt{\mathcal{Z}_\alpha}}} \Psi_\alpha(\mathbf{x}) \pm \Psi_\beta^+(\mathbf{x}) \mathcal{Z}_\beta \]
\[ \hat{\Psi}^{(\pm)}(y) = \sum_{\alpha, \beta} \frac{e^{-i \frac{\mathbf{p}_\alpha \cdot \mathbf{x}}{\hbar}}} {\sqrt{E_\alpha \sqrt{\mathcal{Z}_\alpha}}} \Psi_\alpha(\mathbf{y}) \pm \Psi_\beta^+(\mathbf{y}) \mathcal{Z}_\beta \]

\[ \hat{\Psi}^{(\pm)} \stackrel{\text{annihilation}}{\rightarrow} \hat{\Psi}^{(\pm)} \stackrel{\text{creation}}{\rightarrow} \hat{\Psi}^{(\pm)} \]

\[
\begin{align*}
\{ \Psi_\alpha(x), \Psi_\beta^+(y) \} & = 0 \quad \text{because } \mathcal{Z}_\alpha \Psi_\beta^+(y) = 0, \\
\{ \mathbf{p}, \mathcal{Z}_\beta \} & = 0, \quad \mathcal{Z}_\beta \mathcal{Z}_\beta = 0
\end{align*}
\]
\[
\begin{align*}
\{ \Psi_\alpha^+(x), \Psi_\beta(x) \} & = 0 \quad \text{Similarly} \\
\{ \mathbf{p}, \Psi_\beta(x) \} & = 0 \quad \text{because } \mathcal{Z}_\beta \Psi_\beta(x) = 0 \\
\{ \Psi_\alpha^+(x), \Psi_\beta^+(y) \} & = 0 \quad \text{because } \mathcal{Z}_\beta^2 = 0 \\
\{ \mathbf{p}, \Psi_\beta^+(y) \} & = 0 \quad \text{because } \mathcal{Z}_\beta^2 = 0
\end{align*}
\]

There are only 2 nonzero contractions.

\[ c \left( \Psi_\alpha^+ \circ \Psi_\beta^+ \right) = T \left( \Psi_\alpha^+ \circ \Psi_\beta^+ \right) - N \left[ \Psi_\alpha^+ \circ \Psi_\beta^+ \right] \]

\[ \text{If } t_x > t_y \text{ then} \]
\[ c = \Psi_\alpha^+ \circ \Psi_\beta^+ = i G_{\alpha \beta}^0 (x, y) \text{ for } t_x > t_y \text{ (homework)} \]

\[ \text{If } t_x < t_y \text{ then} \]
\[ c = -\Psi_\alpha^+ \circ \Psi_\beta^+ = 0 \text{ for } t_x < t_y \]

Result
\[ c ( \Psi_\alpha^+ \circ \Psi_\beta^+ ) = \begin{cases} 
 i G_{\alpha \beta}^0 (x, y) \text{ for } t_x > t_y \\
 0 \text{ for } t_x < t_y 
\end{cases} \]

and similarly
\[ c ( \Psi_\alpha^+ \circ \Psi_\beta^+ ) = \begin{cases} 
 0 \text{ for } t_x > t_y \\
 i G_{\alpha \beta}^0 (x, y) \text{ for } t_x < t_y 
\end{cases} \]

Therefore
\[ c ( \Psi_\alpha^+ \circ \Psi_\beta^+ ) = i G_{\alpha \beta}^0 (x, y) \quad \text{Eq. (8.29)} \]

The Wick contraction is equal to the propagator function.
Feynman diagrams

Calculate the Green’s function in perturbation theory.

\[ iG_{\alpha\beta}(x, y) = \langle \Phi_0 | T [\psi_\alpha(x) \psi_\beta(\dagger)(y)] | \Phi_0 \rangle \] (H. picture)

\[ x \text{ means } (t_x, x) \text{ and } y \text{ means } (t_y, y). \]

\[ = \sum_{\nu=0}^{\infty} (-i/\hbar)^\nu \frac{1}{\nu!} \int dt_1 dt_2 \ldots dt_\nu \]

\[ <0|T [H_1(t_1) H_1(t_2) \ldots H_1(t_\nu) \psi_\alpha(x) \psi_\beta(\dagger)(y)] |0> \] (Int. picture)

\[ = \sum_{\nu=0}^{\infty} (-i/\hbar)^\nu \frac{1}{\nu!} \int dt_1 dt_2 \ldots dt_\nu \]

\{ the sum of all complete contractions \}

i.e., all Feynman diagrams

Homework Problem due Friday, Feb 19

Problem 21.

The Wick contraction of \( \psi_\alpha(x) \) and \( \psi_\beta(\dagger)(y) \) is, by Wick’s theorem,

\[ c(\psi_\alpha(x), \psi_\beta(\dagger)(y)) \]

\[ = T[\psi_\alpha(x) \psi_\beta(\dagger)(y)] - N[\psi_\alpha(x)\psi_\beta(\dagger)(y)] \]

Prove:

\[ c(\psi_\alpha(x), \psi_\beta(\dagger)(y)) = i G_{\alpha\beta}(x, y) \]