Electron-electron scattering, in the LO of perturbation theory, at low energies.

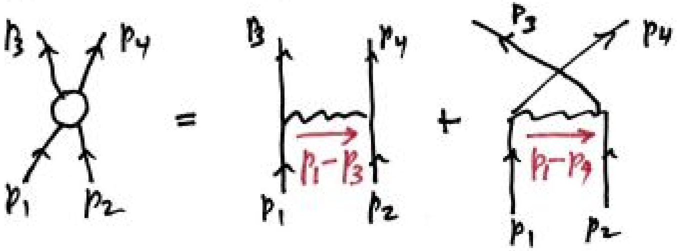
$$\mathcal{M} = \frac{-i}{\Omega} I(\Delta\varepsilon, T)\, \delta_{Kr}(\vec{p}_3+\vec{p}_4 ; \vec{p}_1+\vec{p}_2)$$

$$\left\{ \frac{4\pi e^2}{(\vec{p}_1-\vec{p}_3)^2} A - \frac{4\pi e^2}{(\vec{p}_2-\vec{p}_3)^2} B \right\}$$

$$A = u_3^\dagger u_1 \; u_4^\dagger u_2 = \delta(s_3, s_1)\,\delta(s_4, s_2)$$

$$B = u_3^\dagger u_2 \; u_4^\dagger u_1 = \delta(s_3, s_2)\,\delta(s_4, s_1)$$

In relativistic QFT this comes from 2 Feynman diagrams :



Next : Calculate the scattering cross section .

/6/ The transition probability

The transition probability for i $\longrightarrow$ f is

$$\delta P_{fi} = |\langle f \,|\, i \rangle|^2 \approx |\mathfrak{M}|^2$$

(Born approximation)

So next we need to calculate

$$\delta P = |\mathfrak{M}|^2 \,.$$

$$= \frac{I^2}{\Omega^2} (4\pi e^2)^2 \, \delta_{Kr}(\vec{p}_f, \vec{p}_i) \times$$
$$\times \left\{ \frac{A^2}{|\vec{p}_1 - \vec{p}_3|^4} + \frac{B^2}{|\vec{p}_1 - \vec{p}_4|^4} - \frac{2AB}{(\vec{p}_1 - \vec{p}_3)^2 (\vec{p}_1 - \vec{p}_4)^2} \right\}$$

/7/ Unpolarized scattering

Sum over the final spins (s_3 and s_4);
average over the initial spins (s_1 and s_2).

$$A^2 = \delta_K(s_3, s_1)\, \delta_K(s_4, s_2)$$

$$\overline{A^2} = \frac{1}{2}\sum_{s_1} \frac{1}{2}\sum_{s_2} \sum_{s_3} \sum_{s_4} A^2$$
$$= \frac{1}{4} \sum_{s_1}\sum_{s_3} \delta_K(s_3, s_1) \sum_{s_2}\sum_{s_4} \delta_K(s_4, s_2)$$
$$= \frac{1}{4}\; 2 \;\; 2 \quad = 1$$

$$\overline{B^2} = \frac{1}{4} \sum_{s_1 s_2 s_3 s_4} \delta_K(s_3, s_2)\, \delta_K(s_4, s_1)$$
$$= 1$$

$$\overline{AB} = \frac{1}{4} \sum_{s_1 s_2 s_3 s_4} \delta_K(s_3, s_1)\, \delta_K(s_4, s_2)\, \delta_K(s_3, s_2)\, \delta_K(s_4, s_1)$$
$$= \frac{1}{4} \sum_{s_1 s_2} \delta_K(s_2\, s_1)\, \delta_K(s_2\, s_1)$$
$$= \frac{1}{4}\; 2 \quad = 1/2$$

$$\delta P)_{unpol.} = \frac{T^2}{\Omega^2}(4\pi e^2)^2\, \delta_K(\vec{P}_f, \vec{P}_i) \left\{ \frac{1}{|\vec{p}_1-\vec{p}_3|^4} + \frac{1}{|\vec{p}_1-\vec{p}_4|^4} - \frac{1}{(\vec{p}_1-\vec{p}_3)^2(\vec{p}_1-\vec{p}_4)^2} \right\}$$

/8/ *The scattering cross section, in the center of mass frame of reference*

Recall the definition of cross section:

$$\sigma = \text{transition rate / incident flux}$$

We have transition rate = δP / 2T

(Remember: time $\in$ (–T, T) and eventually we'll take the limit T $\longrightarrow \infty$.)

and

incident flux = # of particles / δA / δt

\# of particles = density * volume

density = 1 / Ω

(one particle in the volume Ω)

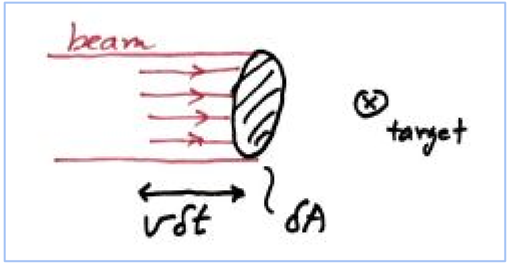
volume = the volume of particles passing area δA in time δt = δA v δt;

in the center of mass frame,

replace v by $v_{relative}$ = v –(–v) = 2 v;

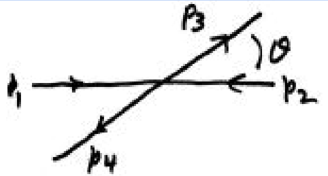
so,

$$\delta\sigma = \frac{\delta P / 2T}{(1/\Omega)(2v\,\delta A\,\delta t)/(\delta A\,\delta t)} = \frac{\Omega\,\delta P}{4\,T\,v}$$



$$d\sigma_{unpol} = \frac{\Omega}{4Tv}\,\frac{I^2}{\Omega^2}(4\pi e^2)^2\,\delta_K(\vec p_3+\vec p_4-\vec p_1-\vec p_2)\left\{\frac{1}{|\vec p_1-\vec p_3|^4}+\frac{1}{|\vec p_1-\vec p_4|^4}-\frac{1}{(\vec p_1-\vec p_3)^2(\vec p_1-\vec p_4)^2}\right\}$$

We'll calculate the center-of-mass cross section



$$\vec p_1 = 0,\ 0,\ p$$
$$\vec p_2 = 0,\ 0,\ -p$$
$$\vec p_3 = p\sin\theta,\ 0,\ p\cos\theta$$
$$\vec p_4 = -p\sin\theta,\ 0,\ -p\cos\theta$$

$$d\sigma_{unpol} = \frac{4\pi\,\delta(\mathcal{E}_f-\mathcal{E}_i)(4\pi e^2)^2}{4v\,4p^4}\,\frac{d^3p_3}{(2\pi)^3}\left\{\frac{1}{(1-\cos\theta)^2}+\frac{1}{(1+\cos\theta)^2}-\frac{1}{(1-\cos\theta)(1+\cos\theta)}\right\}$$

Various ingredients …

$$\sum_{\vec p_4}\delta_K(\vec p_f-\vec p_i)\,(\ldots) = (\ldots)\Big|_{\vec p_4=\vec p_1+\vec p_2-\vec p_3}$$

$$\sum_{\vec p_3} = \frac{\Omega}{(2\pi)^3}\,d^3p_3 \quad \text{(periodic boundary conditions)}$$

$$\frac{1}{T}I^2(\Delta\epsilon,T)\xrightarrow[T\to\infty]{} 4\pi\,\delta(\mathcal{E}_f-\mathcal{E}_i)$$

$$(\vec p_1-\vec p_3)^2 = p^2\sin^2\theta + p^2(1-\cos\theta)^2 = 2p^2(1-\cos\theta)$$

$$(\vec p_1-\vec p_4)^2 = p^2\sin^2\theta + p^2(1+\cos\theta)^2 = 2p^2(1+\cos\theta)$$

We must integrate over $|\vec{p}_3|$,

$$d^3p_3 = p_3^2\, dp_3\, d\Omega_3$$

$$\left(\frac{d\sigma}{d\Omega_3}\right)_{unpol} = \frac{4\pi(4\pi e^2)^2}{16(2\pi)^3}\int \frac{p_3^2\, dp_3}{v p^4}\,\delta(\mathcal{E}_f - \mathcal{E}_i)\ \{\theta \text{ dependence}\}$$

The energy integral: *evaluate carefully!*

$$v = \frac{p}{m} \quad \text{and} \quad \mathcal{E}_i = \frac{p^2}{2m}\times 2$$

$$\text{and} \quad \mathcal{E}_f = \frac{p_3^2}{2m}\times 2 \qquad p_3^2 = m\mathcal{E}_f$$

$$\int \frac{p_3^2\, dp_3}{v p^4}\,\delta(\mathcal{E}_f - \mathcal{E}_i') = \int \frac{(m\mathcal{E}_f)^{1/2}}{(p/m)\,p^4}\,\frac{1}{2} m\, d\mathcal{E}_f\ \delta(\mathcal{E}_f - \mathcal{E}_i')$$

$$= \frac{p m}{2\,(p/m)\,p^4} = \frac{m^2}{2p^4}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{e^4}{2}\,\frac{m^2}{2p^4}\ \{\theta \text{ dependence}\}$$

$$p^2 = \mathcal{E}_i\, m$$

$$= \frac{e^4}{4\,\mathcal{E}_i^2}\left\{\frac{1}{(1-\cos\theta)^2} + \frac{1}{(1+\cos\theta)^2} - \frac{1}{1-\cos^2\theta}\right\}$$

$\mathcal{E}_i = \mathcal{E}_1 + \mathcal{E}_2$ = Total energy in the center of mass frame.

These calculations were done with h-bar=1.
Now we need to restore the factor of h-bar to get the correct units.
e^4 / E^2 has units of area, so the factor of h-bar is (h-bar)^0; i.e., no h-bar factor needed.

The cross section for e e scattering, at low energies, in the center of mass frame of reference is

$(d\sigma/d\Omega)_{\text{unpol.}}$ =

$$\frac{e^4}{4\,E^2}\left\{\frac{1}{(1-\cos\theta)^2}+\frac{1}{(1+\cos\theta)^2}-\frac{1}{1-\cos^2\theta}\right\}$$

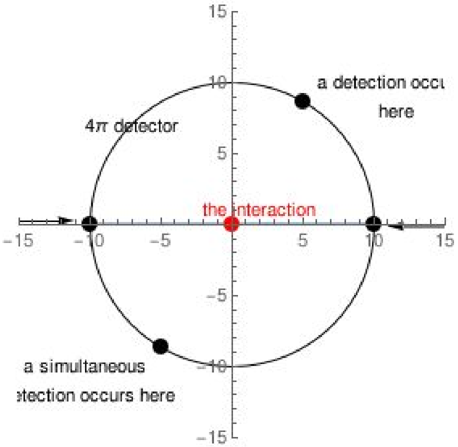
E = total initial kinetic energy in the center of mass frame;
$E = 2 * p^2 /(2m)$

Recall the Rutherford cross section (SI units)

$$\frac{d\sigma}{d\Omega}=\left(\frac{Z_1Z_2e^2}{8\pi\epsilon_0 m v_0^2}\right)^2\csc^4\left(\frac{\Theta}{2}\right). \quad ;$$

i.e., $e^4/(16\,E_1^{\,2})\ * 4 / (1-\cos\theta)^2$ in Gaussian units, with $Z_1=Z_2=1$. (lab frame)

- The cross section for ee scattering is a symmetric function of θ—*symmetric about $\theta = \pi/2$*—because the two particles are identical.
- Note the ***destructive interference*** !


a detection occu
here
4π detector
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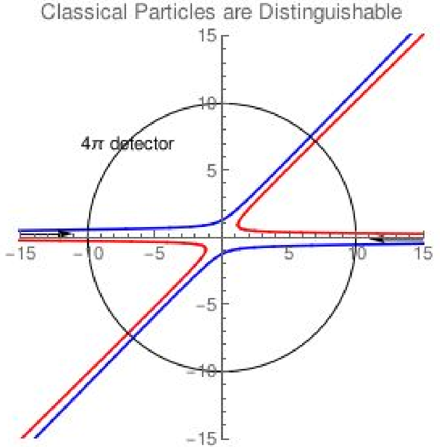
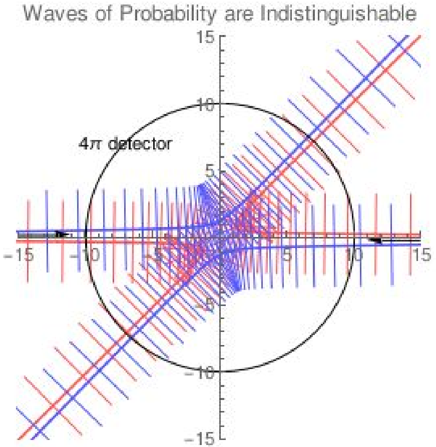
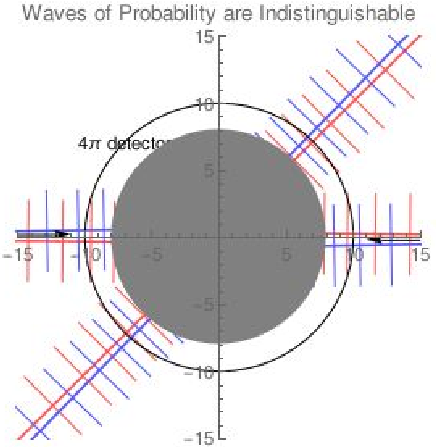

Classical Particles are Distinguishable
4π detector


Waves of Probability are Indistinguishable
4π detector


Waves of Probability are Indistinguishable
4π detecto

Moller scattering cross section from *relativistic* Q.E.D.

$$\frac{d\sigma}{d\Omega} = a_0(a_1 + a_2 + a_3 a_4)$$

$$a_0 = \frac{\alpha^2(2E^2 - m^2)^2}{4E^2(E^2 - m^2)^2}$$

$$a_1 = \frac{4}{\sin^4\theta}$$

$$a_2 = -\frac{3}{\sin^2\theta}$$

$$a_3 = \frac{(E^2 - m^2)^2}{(2E^2 - m^2)^2}$$

$$a_4 = 1 + \frac{4}{\sin^2\theta}$$

In these equations,

- α = the fine structure constant
- E = the total (*relativistic*!) energy in the center of mass frame (E_1+E_2)
- $\hbar = 1$ and $c = 1$
- θ = the center of mass scattering angle

Homework Problem due Friday, February 19

Problem 23.

Use computer graphics.

(a) Plot the Møller cross section $d\sigma/d\Omega(E,\theta)$ as a function of θ, for E = 1.05 MeV, 1.2 MeV and 2.0 MeV. Here E is the *total relativistic energy* in the center of mass frame; θ = the center of mass scattering angle. Put all three functions on the same plot by making a logarithmic plot.

(b) Similarly, plot the low-energy approximation that was derived in class, for the same three values of E. Does the low-energy approximation agree with the Møller cross section at low energies?

Plot θ in degrees from 0 to 180. Plot $d\sigma/d\Omega$ in mb (millibarns). Use a logarithmic axis for the cross section. Use an appropriate range for the vertical axis.

m_e = 0.511 MeV/c^2
0.511+0.511=1.022