LAGRANGIAN FIELD THEORY AND CANONICAL QUANTIZATION **(CHAPTER 2)**

In the history of science, the first field theory was electromagnetism. (Maxwell)

There are 2 vector fields, **E** and **B**.

In spacetime we have a field tensor.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Or, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

where $\partial^\mu = g^{\mu\nu} \partial / \partial x^\nu$

- The *classical field theory* describes electromagnetic waves with $\omega = ck$.
- The *quantum field theory* describes photons. (Chapter 1)
- We can derive the theory from a Lagrangian, and then quantize it.

But there are some subtleties, due to gauge invariance! (Chapter 5)

Electromagnetism isn't very interesting without sources , i.e., *charges*.

- We'll add the electron field in PHY 955.

That's **Q**uantum **E**lectro**D**ynamics. (Ch. 7)

Recall the example of the Schroedinger equation

Classical field theory: $\psi(\mathbf{x},t)$ is a complex function.

$$A = \int_{t_1}^{t_2} dt \int d^3x \left\{ \frac{-i\hbar}{2}\left(\frac{\partial \psi^*}{\partial t}\psi - \psi^* \frac{\partial \psi}{\partial t} \right) - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - V\psi^*\psi \right\}$$

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi}$$

Quantum: $\psi(\mathbf{x},t)$ is a non-hermitian operator.

$$[\psi(x), \psi^+(x')] = \delta^3(x - x')$$
$$[\psi(x), \psi(x')] = 0$$

Now another example: (SECTION 2.2 - 2.3)

A REAL SCALAR FIELD $\varphi = \varphi(\mathbf{x},t)$

This example is relativistically covariant.

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{c^2}{2}(\nabla\phi)^2 - \frac{1}{2}\left(\frac{mc^2}{\hbar}\right)^2\phi^2$$

Derive the field equation from Hamilton's principle,

$$A = \int \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2}(\nabla\phi)^2 - \frac{1}{2}\left(\frac{mc^2}{\hbar}\right)^2\phi^2 \right\} d^3x\, dt$$

$$\delta A = \int \left\{ \dot{\phi}\,(\delta\dot{\phi}) - c^2 \nabla\phi \cdot \nabla(\delta\phi) - \left(\frac{mc^2}{\hbar}\right)^2 \phi\, \delta\phi \right\} d^3x\, dt$$

$$= \int \delta\phi \left\{ -\ddot{\phi} + c^2\nabla^2\phi - \left(\frac{mc^2}{\hbar}\right)^2\phi \right\} d^3x\, dt$$

$= 0$ for any variation $\delta\phi$. Therefore

$$\ddot{\phi} - c^2\nabla^2\phi + \left(\frac{mc^2}{\hbar}\right)^2\phi = 0$$

the Klein Gordon equation

We can solve the Klein-Gordon equation, in plane waves,

$$\phi(\vec{x},t) = C e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

where

$$-\omega^2 + c^2 k^2 + \left(\frac{mc^2}{\hbar}\right)^2 = 0$$

$$\omega = \pm\sqrt{c^2k^2 + m^2c^4/\hbar^2}$$

I.e.,

$$\hbar\omega = \pm\sqrt{c^2\hbar^2k^2 + m^2c^4}$$

Note that this is the energy ($\hbar\omega$) and momentum ($\hbar\mathbf{k}$) relation of special relativity.

(What are the negative energy solutions?)

The general solution (**Hermitian**) is

$$\phi(\vec{x},t) = \sum_{\vec{k}} N \left\{ e^{i(\vec{k}\cdot\vec{x} - \omega t)} a_{\vec{k}} + e^{-i(\vec{k}\cdot\vec{x} - \omega t)} a^{+}_{\vec{k}} \right\}$$

Quantization

We can anticipate

$$[\, a_{\mathbf{k}} , a_{\mathbf{k}'}^{\dagger} \,] = \delta_K(\, \mathbf{k} , \mathbf{k}' \,)$$

$$[\, a_{\mathbf{k}} , a_{\mathbf{k}'} \,] = 0$$

Derive this from Dirac's canonical quantization. Recall,

$$[\, q , p \,] = i\,\hbar \quad \text{where} \quad p = \partial L/\partial q'$$

$$\Pi(\vec{x}) = \frac{\delta L}{\delta \dot{\phi}(x)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x)$$

The E.T.C.R. should be

$$[\phi(\vec{x},t), \Pi(\vec{x}',t)] = i\hbar\, \delta^3(\vec{x}-\vec{x}')$$

$$\phi(x t) = \sum_k N\{a_k e^{i(k\cdot x-\omega t)} + a_k^+ e^{-i(k\cdot x-\omega t)}\}$$

$$\Pi(x,t) = \sum_k N(-i\omega)\{a_k e^{i(k\cdot x-\omega t)} - a_k^+ e^{-i(k\cdot x-\omega t)}\}$$

$$[\phi(x), \Pi(x')] = \sum_k \sum_{k'} NN'(-i\omega') [a_k e^{ik\cdot x} + a_k^+ e^{-ik\cdot x}, a_{k'} e^{ik'x'} - a_{k'}^+ e^{-ik'x'}]$$

$$= \sum_k \sum_{k'} NN'(-i\omega')\{-\delta(kk') e^{ik\cdot(x-x')} - \delta(k,k') e^{-ik\cdot(x-x')}\}$$

$$= \sum_k N^2(i\omega)\{e^{ik\cdot(x-x')} + e^{-ik\cdot(x-x')}\}$$

= should be $i\hbar\,\delta^3(\vec{x}-\vec{x}') = \frac{i\hbar}{(2\pi)^3}\int d^3k\, e^{ik\cdot(x-x')}$

Infinite volume limit $\sum_k = \frac{\Omega}{(2\pi)^3}\int d^3k$

$= i\hbar\,\delta^3(\vec{x}-\vec{x}')$ provided that $N^2 \omega\, 2\Omega = \hbar$

$$N = \sqrt{\frac{\hbar}{2\omega\Omega}}$$

The Hamiltonian

$$H = p\dot{q} - L$$

rewritten in terms of q, p

$$H = \int \Pi(\vec{x})\, \dot{\phi}(\vec{x})\, d^3x - L$$

rewritten in terms of $\phi(\vec{x}), \Pi(\vec{x})$

Homework problem.

(A) Write H in terms of $\pi(\mathbf{x})$ and $\varphi(\mathbf{x})$.

(B) Write H in terms of a_k and $a_k^\dagger$.

Homework problem.

Determine the Green's function for the free scalar field; $\langle 0|\, T\, \varphi(x)\, \varphi(y)\, |0\rangle$.

Next: A real scalar field φ with a source ρ .

To make it simpler, set ħ = 1 and c = 1 . (natural units) At the end of a calculation we can restore the factors of ħ and c by dimensional analysis (i.e., simple units analysis).

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \rho\varphi$$

Field equation $\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta\dot\phi}\right) - \frac{\delta L}{\delta\phi} = 0$

$$\hookrightarrow \quad \ddot\phi - \nabla^2\phi + m^2\phi - \rho = 0$$

The field equation is a linear inhomogeneous equation; so $\varphi(x,t) = \varphi_{particular}(x,t) + \varphi_{homogeneous}(x,t)$.

The particular solution comes from the source; e.g., it could be a mean field produced by a static source; or, waves radiated by a time dependent source.

The homogeneous solution consists of harmonic waves.

The particular solution for a static source

Consider $\rho = \rho(\mathbf{x})$, independent of t

$$-\nabla^2\phi_0 + m^2\phi_0 = \rho(\vec{x})$$

We need the Green's function of $-\nabla^2 + m^2$; i.e.,

$$(-\nabla^2 + m^2)\, G(\vec{x}-\vec{y}) = \delta^3(\vec{x}-\vec{y})$$

Then

$$\phi_0(\vec{x}) = \int G(\vec{x}-\vec{y})\, \rho(\mathbf{y})\, d^3y$$

The Green's function of $-\nabla^2 + m^2$

$$(-\nabla^2+m^2)\,G(\vec{\xi}) = \delta^3(\vec{\xi})$$

$$G(\vec{\xi}) = \int \frac{d^3k}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot\vec{\xi}}}{k^2+m^2}$$

$$= \frac{1}{(2\pi)^3}\cdot 2\pi \int_0^\infty \frac{k^2\,dk}{k^2+m^2}\underbrace{\int_{-1}^{1} d\cos\theta\; e^{ik\xi\cos\theta}}_{\frac{1}{ik\xi}\left(e^{ik\xi}-e^{-ik\xi}\right)}$$

$$= \frac{1}{4\pi^2 i\xi}\int_{-\infty}^{\infty}\frac{k\,dk}{(k-im)(k+im)}\,e^{ik\xi}$$

$\xi > 0$ so close the contour above

$$= \frac{1}{4\pi^2 i\xi}\;\frac{2\pi i\,(im)}{2im}\;e^{-m\xi}$$

$$= \frac{e^{-m|\vec{\xi}|}}{4\pi|\vec{\xi}|} \qquad (\text{w/ } \hbar=1 \text{ and } c=1)$$

Example

Suppose $\rho(\boldsymbol{x}) = \rho_o\,\theta(a - r)$.

$$\phi_0(\vec{x}) = \int \frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}\,\rho(\vec{y})\,d^3y$$

$$\phi_0(r) = \frac{\rho_0}{4\pi}\int \frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\,\theta(a-y)\,d^3y$$

Limiting cases —

- Large r

$$\phi_0(r) \sim \frac{\rho_0}{4\pi}\,\frac{e^{-mr}}{r}\cdot\frac{4}{3}\pi a^3$$

- Small r

$$\phi_0(r) \sim \frac{\rho_0}{4\pi}\int \frac{e^{-my}}{y}\,\theta(a-y)\,d^3y$$

$$= \frac{\rho_0}{m^2}\left\{1-(1+ma)\,e^{-ma}\right\}$$

Still $\hbar=1$ and $c=1$.

The interaction Lagrangian density

$$\mathfrak{L}_{\text{interaction}} = g\ \Psi^{\dagger}{}_{\alpha\rho}\ \Psi_{\alpha\rho}\ \varphi$$

- This $\mathfrak{L}_{\text{int}}$ acts as a source for φ, with

  $\rho\,(\mathbf{x},t) = g\ \Psi^{\dagger}{}_{\alpha\rho}\ \Psi_{\alpha\rho}$.

- It also acts as a potential for Ψ :

  $V_{\text{int}}(x,t) = -\,g\ \varphi(x,t)$.

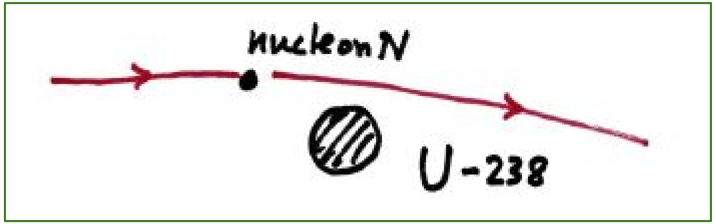
∴ The field equations;

i.e., Lagrange's equations,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + (-g\phi)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = g\,\psi^{\dagger}\psi$$

Calculate the potential energy for a nucleon (N) attracted to a heavy isotope (Z,A)



First step -- calculate the mean field created by the nucleons in the heavy isotope.

$$-\nabla^2\phi + m^2\phi = \langle g\,\psi^\dagger\psi\rangle$$

$$\langle\psi^\dagger\psi\rangle_{U-238} = \text{density of nucleons}$$

$$= \sum_{n=1}^{238} |u_n(\vec{x})|^2$$

$$\approx \frac{A}{\frac{4}{3}\pi R^3}\,\theta(R-r) \text{ where } R = r_0 A^{1/3}$$

$$\phi(\vec{x}) = \int G(\vec{x}-\vec{y})\; n(\vec{y})\; d^3y$$

Second step -- calculate the potential energy for the presence of the extra nucleon.

$$V(\vec{x}) = -g\phi = -g\int G(x-y)\,n(y)\,d^3y$$

$$V(\vec{x}) = \frac{-3g}{4\pi r_0^3}\frac{1}{A}\int \frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}\,\theta\left(r_0A^{1/3}-|\vec{y}|\right)d^3y$$

***ħ* = 1 and *c* = 1 .**

Rewrite this for numerical calculation...

Yukawa's theory of the nucleon-nucleon force (1935)

(1) Nucleons interact through a scalar field φ with mass m.

(2) The range of the force is

$$\text{range} = \frac{\hbar}{mc} = 1 \text{ to } 2 \text{ fm}$$

$$\therefore\ mc^2 = \frac{\hbar c}{\text{range}} = 100 \text{ to } 200 \text{ MeV}$$

Of course Yukawa did not know about pions, which were discovered in 1947.

mass ($\pi^{\pm}$) = 139.6 MeV/c^2
mass (π^{0}) = 135.0 MeV/c^2

The Lagrangian density for the theory is

$$\mathfrak{L} = \mathfrak{L}_{\text{nucleon}} + \mathfrak{L}_{\text{meson}} + \mathfrak{L}_{\text{interaction}}$$

Nucleon field = $\psi_{\alpha\rho}(\vec{x})$

α = spin index and ρ = isospin index

$$\psi_{\alpha\rho}(\vec{x}) = \sum_{\vec{k},s,t} \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \eta_\alpha^{(s)} \eta_\rho^{(t)} a_{\vec{k}st}$$

Meson field = $\phi(\vec{x})$
with isospin 0 to follow Yukawa

Lagrange's equations including the interaction,
$\mathfrak{L}_{\text{interaction}} = g\, \psi^{\dagger}_{\alpha\rho} \psi_{\alpha\rho}\, \varphi$:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + (-g\phi)\psi = i\hbar \frac{\partial \psi}{\partial t}$$

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi + m^2\phi = g\,\psi^{\dagger}\psi$$

Numerical calculations

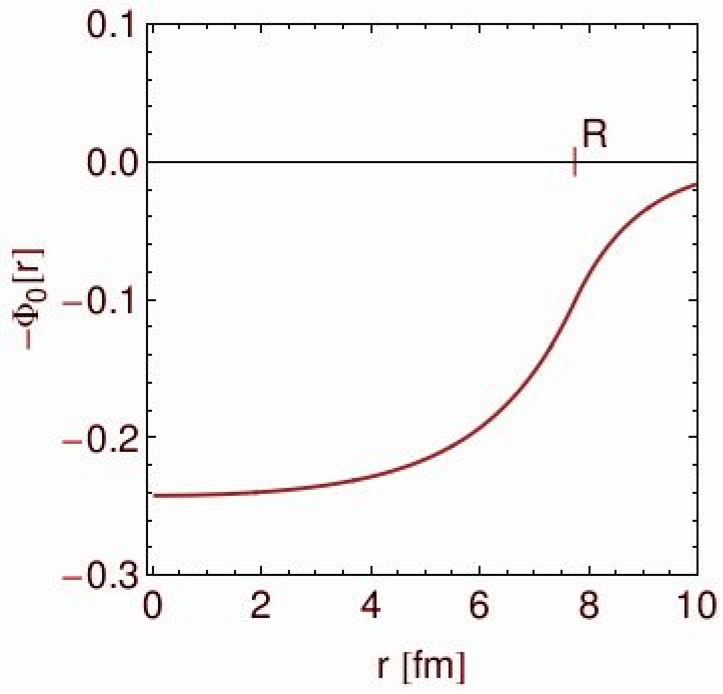
r_0 = 1.25 fm — $R = r_o A^{1/3}$

mc^2 = 140 MeV — pion mass

A = 238 — uranium

g = 15 — strong interaction

The potential energy for the extra nucleon is $V(r) = -\,g^2\,\Phi_0(r)$.



Homework due Wednesday, March 2

Problem 32.

For the free real scalar field,

(A) Write H in terms of $\pi(\mathbf{x})$ and $\varphi(\mathbf{x})$.

(B) Write H in terms of a_k and $a_k^\dagger$.

Problem 33.

(A) Mandl and Shaw problem 3.3.

(B) Mandl and Shaw problem 3.4.