The Theory of the Photon

Planck, Einstein → the origins of quantum theory

Dirac

↳ the first quantum field theory

The equations of electromagnetism in empty space
(i.e., no charged particles are present)
using gaussian units,

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$U = \frac{1}{8\pi}\int \left(E^2 + B^2\right) d^3x$$

We can solve the equations of electromagnetism, by introducing potentials φ($\mathbf{x}$,t) and $\mathbf{A}$($\mathbf{x}$,t) ...

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \quad \text{and} \quad \vec{B} = \nabla\times\vec{A}$$

Then

$$\nabla\cdot\vec{E} = \frac{-1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{A}) - \nabla^2\phi = 0 \qquad \text{is required;}$$

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \frac{-1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{A}) = 0$$

is automatic;

$$\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0 \qquad \text{is automatic;}$$

$$\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi) = 0$$

is required

W.L.O.G. we can require $\nabla \cdot \mathbf{A} = 0$,
which is called the *Coulomb gauge condition*.

Then $\Phi = 0$ and

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = 0 \quad \text{(wave equation)}$$

Thus, ***A*(*x*,*t*)** *is an infinite set of harmonic oscillators.*

Or, I should say these are *coupled* oscillators.

$$\ddot{q} + \omega^2 q = 0$$

$$\ddot{q}_i + \sum_j K_{ij} q_j = 0$$

$$i \rightarrow \vec{x} \quad \text{and} \quad q_i \rightarrow \vec{A}(\vec{x})$$

And how do we calculate coupled oscillations?

We use the *normal modes*.

“Normal modes” of the electromagnetic field

The normal modes of the electromagnetic field are transverse plane waves.

Try $\mathbf{A}(\mathbf{x},t) = C\,\boldsymbol{\varepsilon}\,e^{i(\mathbf{k}.\mathbf{x}-\omega t)}$ (**ε** is a unit vector)

The Coulomb gauge condition requires $\mathbf{k}\cdot\boldsymbol{\varepsilon} = 0$;
ie, the waves are transverse.

The wave equation requires $\omega = c\,|\mathbf{k}|$, which is called the *dispersion relation*.

The wave velocity is

$$\frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k} = c$$

(This is the starting point for the theory of special relativity.)

“Normal modes”; or, better, plane wave solutions

$$\vec{\varepsilon}\cdot\vec{k} = 0$$

$$v_{phase} = \frac{\omega}{k} = c$$

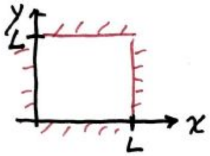
The general solution of the field equation is a superposition of ~~normal modes~~ *plane waves*,

$$\vec{A}(\vec{x},t) = \sum_{\vec{k}} \sum_{\sigma=1}^{2} \hat{\varepsilon}_\sigma \left[C(\vec{k},\sigma)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} + C^*(\vec{k},\sigma)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \right]$$

where $\omega = ck$

and $\vec{k}$, $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ form an orthogonal triad.

__Infinite space, or finite?__ For normalizability, we use a finite space with dimensions L x L x L, with periodic boundary conditions, and then take the limit L$\rightarrow$ infinity.



$$\vec{A}(\vec{x}+L\hat{e}_i, t) = \vec{A}(\vec{x}, t)$$

Then for the plane waves,

$\mathbf{k} = (2\pi/L)\,\mathbf{n}$ where $\mathbf{n}$ is integer valued.

Volume $\Omega = L^3$.

Infinite volume limit : $\sum_{\mathbf{k}} \longrightarrow \Omega \int d^3k \, /(2\pi)^3$

So far, we have only discussed the wave solutions of the field equations. *Where are the photons?*

Following Dirac, we replace the c-numbers ,

$c(\mathbf{k},\sigma)$ and $c^*(\mathbf{k},\sigma)$,

by annihilation and creation operators,

$$c(\mathbf{k},\sigma) \to a_{\mathbf{k}\sigma} \qquad \text{and} \qquad c^*(\mathbf{k},\sigma) \to a^{+}_{\mathbf{k}\sigma}$$

with

$$[\, a_{\mathbf{k}\sigma} \,,\, a^{+}_{\mathbf{k}\sigma} \,] = 1$$

$$[\, a_{\mathbf{k}\sigma} \,,\, a^{+}_{\mathbf{k}'\sigma'} \,] = 0 \quad \text{for} \;\; \{\mathbf{k}',\sigma'\} \neq \{\mathbf{k},\sigma\}$$

i.e., $$[\, \mathbf{a}_{\mathbf{k}\sigma} \,,\, \mathbf{a}^{+}_{\mathbf{k}'\sigma'} \,] = \delta_{\mathbf{k},\mathbf{k}'}\, \delta_{\sigma\sigma'}$$

All other commutators are 0; e.g., $[a_{\mathrm{k}\sigma} \,,\, a_{\mathrm{k}'\sigma'}] = 0$

So we write

$$\mathbf{A}(\mathbf{x},t) = \sum N_k\, \boldsymbol{\varepsilon}_{\mathbf{k}\sigma} \{ a_{\mathbf{k}\sigma}\; e^{i(\mathbf{k}.\mathbf{x} - \omega t)} + a^{+}_{\mathbf{k}\sigma}\; e^{-i(\mathbf{k}.\mathbf{x} - \omega t)} \}$$

and that is the quantized electromagnetic field,

in the Heisenberg picture.

If this is to be a quantum theory, there must be a Hamiltonian (= the generator of translation in time).

What is the Hamiltonian? (Homework Problem 3)

<u>The normalization factor</u> N_k

$$N_k = \mathrm{Sqrt}[\, 2\pi\, \hbar\, c^2 \;/\; \omega\Omega]$$ (remember, ω=ck)

which is necessary so that $H = 1/(8\pi) \int (E^2 + B^2)\, d^3x$.

Pictures

Here is something you must understand for quantum field theory. You have already studied it, but we'll go over it again.

- All predictions in quantum theory are based on matrix elements; e.g., $O_{\alpha\beta} = \langle \alpha \,|\, O \,|\, \beta \rangle$.
- Matrix elements may depend on time. But what depends on time—the states or the observables?
- In the Schroedinger picture, the states depend on time, but the observables do not depend on time.
- In the Heisenberg picture, the states do not depend on time, but the observables depend on time.
- *Essential:* $\langle \alpha,t \,|\, O \,|\, \beta,t \rangle_{\text{Schr.}} = \langle \alpha |\, O(t) \,| \beta \rangle_{\text{Heis.}}$

(Homework Problem 4)

(Needed for Homework Problem 4)

Time dependence in the Schroedinger picture

$$i \hbar \, (\partial/\partial t) \; | \alpha, t > \; = H \, | \alpha, t > \qquad ;$$

or,

$$| \alpha, t > \; = \exp(- i H t / \hbar) \, | \alpha, 0 > .$$

Time dependence in the Heisenberg picture

$$\exp(+ i H t / \hbar) \, \mathcal{O}(0) \, \exp(- i H t / \hbar) = \mathcal{O}(t) \qquad ;$$

or,

$$- i \hbar \, (\partial/\partial t) \, \mathcal{O}(t) \; = [\; H , \mathcal{O}(t)] \, .$$

Homework due Friday, January 22 …

Problem 3.

The quantum field for the free electromagnetic field $\mathbf{A}(\mathbf{x},t)$ in the Heisenberg picture is written as an expansion in plane waves, with the annihilation and creation operators, $a_{\mathbf{k}\sigma}$ and $a^{\dagger}_{\mathbf{k}\sigma}$.

From that, determine the Hamiltonian H.

Problem 4.

Start with the equations for time dependence in the Schroedinger and Heisenberg pictures.

Prove $\langle \alpha,t \,|\, O \,|\, \beta,t \rangle_{\text{Schr.}} = \langle \alpha |\, O(t) \,| \beta \rangle_{\text{Heis.}}$