Interactions of Radiation and Matter

$$H = H_{atom} + H_{rad} + H_{int}$$

We know

$$H_{atom} = \frac{p^2}{2m} - \frac{Ze^2}{r} \quad \text{(one electron atom or ion)}$$

$$H_{rad} = \sum_{\vec{k}\sigma} \hbar\omega \, a^{\dagger}_{\vec{k}\sigma} a_{\vec{k}\sigma} \quad \text{where} \quad \omega = c\vec{k}$$

But what is H_{int}?

Start with the Hamiltonian for an electron in an electromagnetic field,

$$H = \frac{(\vec{p} + e/c\, \vec{A})^2}{2m} - e\phi$$

(electron charge = -e)

Why?

Use Hamilton's equations,

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)$$

$$\dot{\vec{p}} = \frac{-\partial H}{\partial \vec{x}} = \frac{-1}{m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)_j \frac{e}{c} \frac{\partial A_j}{\partial \vec{x}} + e\, \partial\phi/\partial\vec{x}$$

$$= -\dot{x}_j \frac{e}{c} \frac{\partial A_j}{\partial \vec{x}} + e \nabla\phi$$

Also,

$$\dot{\vec{p}} = m\ddot{\vec{x}} - \frac{e}{c}\dot{\vec{A}}$$

$$= \vec{F} - \frac{e}{c}\left[\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}\right]$$

$$\therefore \vec{F} = -e\left[\frac{\vec{v}}{c} \times \vec{B} + \vec{E}\right]$$ as required

Now write $H = p^2/2m - e\,\varphi(\mathbf{x}) + H_{int}$

So,

$$H_{int} = \frac{e}{mc}\vec{A}(\vec{x})\cdot\vec{p} + \frac{e}{2mc^2}A^2(\vec{x})$$

$$\boxed{\vec{A}(\vec{x}) = \sum_{\vec{k}\sigma} N_{\vec{k}}\,\hat{\epsilon}_{\vec{k}\sigma}\left[a_{\vec{k}\sigma}e^{i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}\sigma}e^{-i\vec{k}\cdot\vec{x}}\right]}$$

(in the Schroedinger picture)

The first term of H_{int} will annihilate or create a photon.
The second term could annihilate or create 2 photons.

$$N_k = \sqrt{\frac{2\pi\hbar c^2}{\omega\Omega}}$$

Radiative decays in atomic physics

$$a \longrightarrow b + \gamma \qquad \text{where} \quad \varepsilon_a > \varepsilon_b$$

We'll calculate the transition rate by Fermi's Golden Rule, from perturbation theory, treating H_{int} as a perturbation.

$$\text{Rate} = \frac{2\pi}{\hbar} \sum_f \left| \langle f | H_{int} | i \rangle \right|^2 \delta(E_f - E_i)$$

(first order approximation in time-dependent perturbation theory).

The initial state is $\quad | i \rangle = | a \rangle_{\text{atom}} \, | \text{vacuum} \rangle_{\text{rad.}}$

The final state is $\quad | f \rangle = | b \rangle_{\text{atom}} \, | \mathbf{k}, \boldsymbol{\varepsilon} \rangle_{\text{rad.}}$

i.e., $a^{+}_{k\varepsilon} \, | \text{vacuum} \rangle_{\text{rad.}}$

Calculate the transition matrix element

$\langle f | H_{int} | i \rangle =$

$$\langle b | \langle \vec{k}\vec{\epsilon} | \frac{e}{mc} \vec{A} \cdot \vec{p} \, | \text{vacuum} \rangle | a \rangle$$

$$= \frac{e}{mc} \langle b | \underbrace{\langle \vec{k}\vec{\epsilon} | \vec{A} | \text{vac.} \rangle}_{N_k \hat{\epsilon}_{\vec{k}\sigma} e^{-i\vec{k}\cdot\vec{x}}} \cdot \vec{p} \, | a \rangle$$

$(=\text{coefficient of } a^{+}_{\vec{k}\sigma} \text{ in } \vec{A}(\vec{x}))$

$$= \frac{e}{mc} N \vec{\epsilon} \cdot \langle b | e^{-i\vec{k}\cdot\vec{x}} \, \vec{p} \, | a \rangle$$

The size of the atom is << the wavelength of the photon;

10^{-10} m << 100 nm = 10^{-7} m

so we can approximate $e^{-ik.x} \approx 1$.

$$\text{Rate} = \frac{2\pi}{\hbar} \sum_{\vec{k}\sigma} \left(\frac{e}{mc}\right)^2 N^2 \left| \vec{\varepsilon}_\sigma \cdot \langle b | \vec{p} | a \rangle \right|^2 \delta(\epsilon_b + \hbar\omega - \epsilon_a)$$

The rest is just math.

Example:

The decay H(2p) → H(1s) + γ for atomic hydrogen.

The atomic matrix element is

$\langle 2p, m \,|\, \mathbf{p} \,|\, 1s \rangle$ for m = 0, +1, −1

This is just a problem in ordinary quantum mechanics.

The calculation is not difficult.

Recall the wave functions,

$\Phi_{1s}(x) = \exp(-r/a_0) \;/(\pi\, a_0^{\,3})^{1/2}$

Let's take m=0 , *(m=1 or -1 would have the same total decay rate)*

$\Phi_{2p}(x) = (r/a_0\;)\;\; \exp(-r\,/2a_0)\;\; \cos\theta \,/(\,32\,\pi\, a_0^{\,3})^{½}$

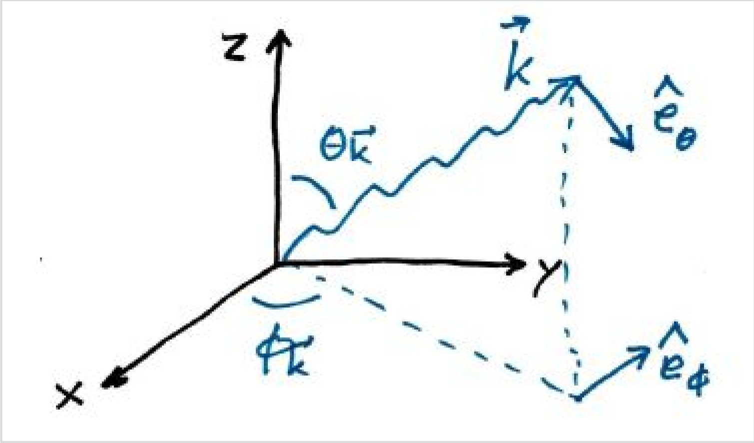
$\langle 2p, 0 \,|\, \mathbf{p} \,|\, 1s \rangle = -i\hbar \int r^2\, dr\, \sin\theta\, d\theta\, d\varphi \;\; \Phi^*_{2p} \nabla \Phi_{1s}$

$= \mathbf{e_z}\, (i\hbar/a_0)\, (⅔)^4\, \sqrt{2}$

Sum over polarizations

To get the unpolarized decay rate we must sum over the final two polarizations , $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$. They must both be perpendicular to $\mathbf{k}$. Let $\boldsymbol{\varepsilon}_1 = \mathbf{e}_\varphi$ and $\boldsymbol{\varepsilon}_2 = \mathbf{e}_\theta$.

Note that $\mathbf{e}_z \cdot \mathbf{e}_\varphi = \mathbf{0}$ and $\mathbf{e}_z \cdot \mathbf{e}_\theta = -\sin\theta_k$



So,

$$\sum_\sigma |\, \boldsymbol{\varepsilon}_\sigma \cdot \langle b|\, \mathbf{p}\, |a\rangle|^2 = (\mathbf{e}_\theta \cdot \mathbf{e}_z)^2\, (\hbar/a_0)^2\, (2^9 / 3^8)$$

$$= \sin^2\theta_k \;\; (\hbar/a_0)^2\, (2^9 / 3^8)$$

and Rate =

$$\frac{2\pi}{\hbar} \sum_{\vec{k}} \left(\frac{e}{mc}\right)^2 N^2 \sin^2\theta_k \left(\frac{\hbar}{a_0}\right)^2 \frac{2^9}{3^8} \times \delta(\epsilon_b + \hbar\omega - \epsilon_a)$$

$$N^2 = \frac{2\pi\hbar c^2}{\omega\Omega}$$

The differential decay rate (for m = 0)

We need to calculate the sum over **k**.

Remember the infinite volume limit,

$$\sum_{\vec{k}} = \frac{\Omega}{(2\pi)^3}\int d^3k$$

(periodic boundary conditions)

and $d^3k = k^2\,dk\,d\Omega_k$

To get the total rate, integrate over all directions of **k**.
To get the differential rate we don't integrate over directions...

$$dR/d\Omega_k = \frac{512}{6561}\,\frac{\hbar^2 c^2}{2\pi a_0^2}\left(\frac{e}{mc}\right)^2 \sin^2\theta_k \int \frac{k^2\,dk}{\omega}\,\delta(\epsilon_b - \hbar\omega - \epsilon_a)$$

$(\omega = ck)$

The k integration is evaluated with the delta function.
Remember how to do delta function integrations:

$$\int_{-\infty}^{\infty} \delta(ax)\ f(x)\ dx = 1/a\ \ f(0)$$

Final results for the decay $H_{2p} \mapsto H_{1s} + \gamma$

$$\frac{dR}{d\Omega_k} = \frac{512}{6561} \frac{\hbar\omega_{ab}}{c a_0^2} \left(\frac{e}{mc}\right)^2 \frac{\sin^2\theta_k}{2\pi}$$

$\propto \sin^2\theta$ because m = 0

$$\frac{dR}{d\Omega_k} = \frac{64}{2187} \frac{mc^2}{\hbar} \alpha^5 \frac{\sin^2\theta_k}{2\pi}$$

$$R = \frac{256}{6561} \frac{mc^2}{\hbar} \alpha^5$$

$2p \to 1s + \gamma$

$$\hbar\omega_{ab} = \frac{3}{4} \frac{me^4}{2\hbar^2}$$

$a_0 = \hbar^2/me^2$ and $\alpha = e^2/\hbar c$

Homework due Friday, January 29

Problem 7. Calculate the mean lifetime of the 2p state of atomic hydrogen.

Problem 8. Estimate the mean lifetime for a nuclear gamma decay process. (You only need to calculate the order of magnitude.)