Light Scattering by Electrons

This subject has an interesting history.

§ Rayleigh scattering (1871)
  classical theory,
    pre-electron discovery;
  why the sky is blue.

§ Thomson scattering (1906)
  classical theory,
    post-electron discovery;
  light scattering by plasmas .

§ Compton scattering (1923)
  experimental;
  the failure of the classical wave theory, and a triumph of the photon theory.

§ Dirac (1927)
  quantum field theory;

§ Raman scattering (1928)
  experimental;
  inelastic scattering of light by atoms.

§ Klein-Nishina formula (1928)
  theory;
  for relativistic electrons.

The Klein-Nishina formula

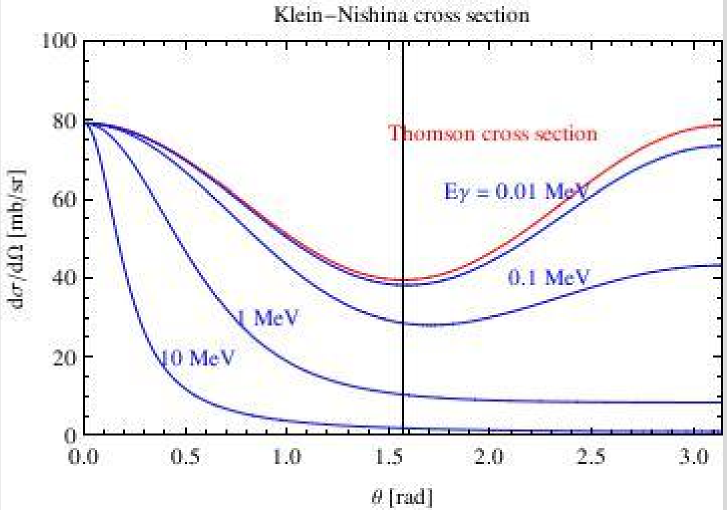
This is the formula for photon-electron scattering,
in first order perturbation theory in relativistic Q.E.D.

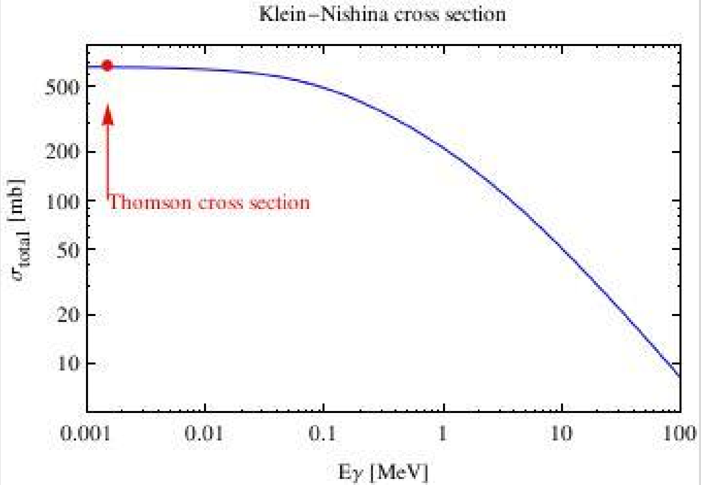
(We'll derive it in PHY 955.)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2} \lambda\!\!\!^{-}{}_{C}^{2}\, F^2 \left(F + \frac{1}{F} - \sin^2\theta \right)$$

$$F = F(E_\gamma, \theta) = \frac{mc^2}{mc^2 + E_\gamma (1 - \cos\theta)}$$

where $\alpha = e^2/\hbar c = 1/137$
and $\lambda\!\!\!^{-}{}_{\text{Compton}} = \hbar/mc = 386\ fm$

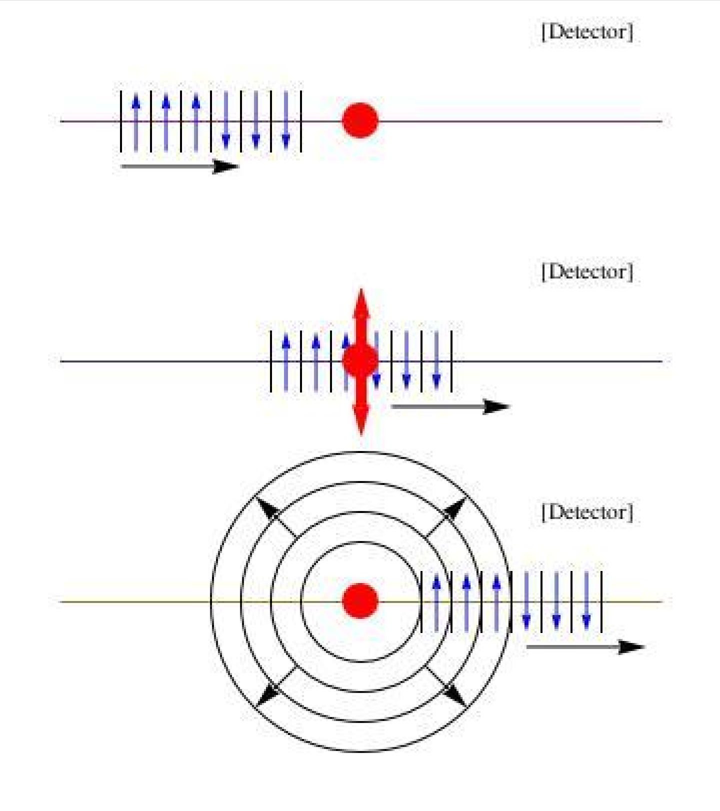




Light scattering by electrons -- sketches

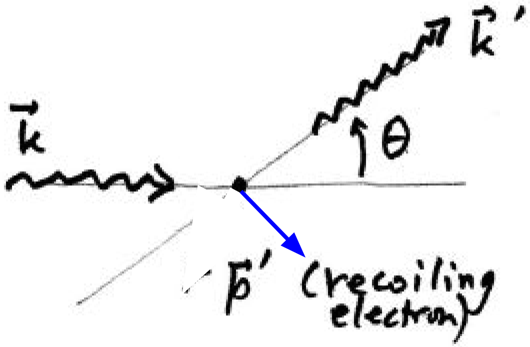
The classical wave theory:
-- the wave fronts represent a physical wave;
they carry energy;
the detector must register some scattered energy.

The photon theory:
-- the wave fronts represent a photon;
they only carry probability;
the total energy is ħω;
the detector will either register energy ħω, or nothing.



Photon scattering by a free electron
-- the kinematics

$$\gamma + e \longrightarrow \gamma' + e'$$



$$\hbar k + p = \hbar k' + p'$$

$$\hbar c k + E_p = \hbar c k' + E_{p'}$$

$$E_p = \sqrt{m^2c^4 + p^2c^2} = mc^2 \text{ in rest frame of } e$$

$$E_{p'} = \sqrt{m^2c^4 + p'^2c^2} = \sqrt{m^2c^4 + \hbar^2c^2(\vec{k} - \vec{k}')^2}$$

in rest frame of e

$$mc^2 + \hbar c(k-k') = \sqrt{m^2c^4 + \hbar^2c^2(\vec{k}-\vec{k}')^2}$$

$$m^2c^4 + \hbar^2c^2(k^2 + k'^2 - 2kk') + 2mc^2\hbar c(k-k')$$

$$= m^2c^4 + \hbar^2c^2(k^2 + k'^2 - 2kk'\cos\theta)$$

$$-\hbar c \cdot 2kk' + 2mc^2(k-k') = \hbar c(-2kk'\cos\theta)$$

$$k' = \frac{mc^2 k}{mc^2 + \hbar c k(1-\cos\theta)} = kF$$

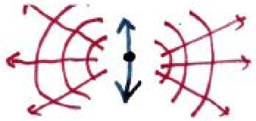
$$\therefore \lambda' = \lambda + \frac{2\pi\hbar}{mc}(1-\cos\theta)$$

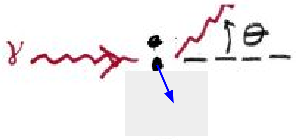
For $E_\gamma \ll mc^2$, $k' \simeq k$; i.e., $F \simeq 1$.

<u>Thomson scattering</u>

<u>= light scattering by a free electron, in the low-energy limit.</u>

Thomson (1906) provided the calculation for the cross section, *using classical wave theory* , valid in the long wavelength limit.





Dirac (1927) rederived the result using quantum field theory.

$$\gamma + e \longrightarrow \gamma' + e' \qquad \text{with} \qquad E_\gamma << mc^2$$

In this limit, we may approximate $E'_\gamma \approx E_\gamma$;
so it was called “elastic scattering”.
The electron does recoil, so $\mathbf{p}' \neq \mathbf{p}$;
however we may approximate $E'_e \approx E_e = m\,c^2$.

The scattering cross section

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{\text{rate}}{\text{incident flux}}$$

We'll use time-dependent perturbation theory.
Recall Fermi's Golden Rule,

Rate = $\frac{2\pi}{\hbar} \sum_f |\langle f|T|i\rangle|^2 \; \delta(\varepsilon_f - \varepsilon_i)$

$| i \rangle = |\mathbf{k},\boldsymbol{\varepsilon}\rangle_\gamma \; |\mathbf{p}\rangle_e$,with $\mathbf{p} = \mathbf{0}$ (rest frame)

$| f \rangle = |\mathbf{k}',\boldsymbol{\varepsilon}'\rangle_\gamma \; |\mathbf{p}'\rangle_e$

$$\boxed{\sum_f = \sum_{\mathbf{k}'} \sum_{\mathbf{p}'} = \int \Omega/(2\pi)^3 \; k'^2\, dk'\, d\Omega'_k \sum_{\mathbf{p}'}}$$

Calculate the transition matrix element

In general, through second order perturbation theory, we would have

$$< f \,|\, T \,|\, i > \; = \; < f \,|\, H_{int} \,|\, i > + \sum_j \frac{< f \,|\, H_{int} \,|\, j >< j \,|\, H_{int} \,|\, i >}{E_i - E_j}$$

Recall

$$H_{int} = \frac{e}{mc} \vec{A} \cdot \vec{p} + \frac{e^2}{2mc^2} A^2$$

Tricky points:

We want the cross section accurate to lowest order in e.

First order term (in H_{int}) is order e^2 from $H^{(2)}_{int}$;

Second order term (in H_{int}) is order e^2 from $H^{(1)}_{int} \times H^{(1)}_{int}$;

but that term is zero because

the initial electron is at rest: $H_{int} \,|\, i > \;\propto\;\; \mathbf{p} \,|\, \mathbf{p} = 0 >_e = 0$.

So, to order e^2 the transition matrix element is just

$$< f \,|\, H^{(2)}_{int} \,|\, i > \;=\; \frac{e^2}{2mc^2} \quad < f \,|\, A^2\,(\mathbf{x}) \,|\, i >$$

Take $a_{\mathbf{k}\boldsymbol{\varepsilon}}$ from one factor of $\mathbf{A}(\mathbf{x})$, and $a^+_{\mathbf{k'}\boldsymbol{\varepsilon}'}$ from the other factor of $\mathbf{A}(\mathbf{x})$; $\therefore$ there are two equal terms .

$$< f \,|\, H^{(2)}_{int} \,|\, i > \;=\; \frac{e^2}{2mc^2} \quad <\mathbf{p}'|\, N_k\, N_{k'}\, \boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'\, e^{\,i\,(\mathbf{k}-\mathbf{k}')\,.\,\mathbf{x}} \;\times 2\, |\mathbf{p}>$$

- The electron matrix element

$$<\mathbf{p}'|\;\; e^{\,i\,(\mathbf{k}-\mathbf{k}')\,.\,\mathbf{x}}\;\; |\mathbf{p}> \;=\; \delta_{Kr.}\,[\,\mathbf{p}',\, \hbar(\mathbf{k}-\mathbf{k}')\,]$$

(p=0)

$$\mathbf{A}(\mathbf{x}) = \Sigma\; N_k\; \boldsymbol{\varepsilon}\,\{\, a_{\mathbf{k}\boldsymbol{\varepsilon}}\;\; e^{\,i\,\mathbf{k}.\mathbf{x}} \;+ a^+_{\mathbf{k}\boldsymbol{\varepsilon}}\;\; e^{\,-i\,\mathbf{k}.\mathbf{x}}\,\}$$

- The electron matrix element

$$\langle \vec{p}\,' | \, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \, | \vec{p} \rangle$$

$$= \int \frac{e^{-i\vec{p}\,'\cdot\vec{x}/\hbar}}{\sqrt{\Omega}} \, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \, \frac{e^{i\vec{p}\cdot\vec{x}/\hbar}}{\sqrt{\Omega}} \, d^3x$$

$$= \frac{1}{\Omega} \int_\Omega e^{i(\vec{p}+\vec{k}-\vec{p}\,'-\vec{k}')\cdot\vec{x}} \, d^3x \qquad \boxed{\hbar = 1}$$

finite volume Ω with periodic boundary conditions

$$= \frac{1}{\Omega} \, \Omega \, \delta_{Kr.}(\vec{p}\,'+\vec{k}', \vec{p}+\vec{k})$$

$\vec{p} = 0$ in lab frame

$$= \delta_{Kr.}(\vec{p}\,', \hbar(\vec{k}-\vec{k}'))$$

The Thomson cross section for *polarized scattering;*
i.e., suppose the initial photon is polarized and the polarization of the final photon is measured. Then the cross section depends on **ε** and **ε'**.

$$\frac{dR}{d\Omega_{k'}} = \underbrace{\frac{2\pi}{\hbar}}_{\frac{2\pi}{\hbar}} \left(\frac{e^2}{mc^2}\right)^2 \underbrace{\left(\sqrt{\frac{2\pi\hbar c^2}{\omega\Omega}}\right)^4}_{\frac{(2\pi)^2\hbar^2 c^4}{\omega^2\Omega^2}} (\vec{\epsilon}\cdot\vec{\epsilon}\,')^2 \underbrace{\frac{\Omega}{(2\pi)^3}}_{\frac{\Omega}{(2\pi)^3}} \underbrace{\frac{k^2}{\hbar c}}_{\frac{\omega^2}{\hbar c^3}}$$

$$= \frac{c}{\Omega} = \text{the incident flux}$$

(incident flux = density × velocity = $\frac{1}{\Omega}\cdot c$)

$$\text{rate} = \frac{2\pi}{\hbar} \sum_f \left(\frac{e^2}{2mc^2}\right)^2 N_k^2 N_{k'}^2 (\vec{\epsilon}\cdot\vec{\epsilon}\,')^2\, 4$$

$$\delta_{Kr}[\vec{p}\,', \hbar(\vec{k}-\vec{k}')]\ \delta(E_f - E_i)$$

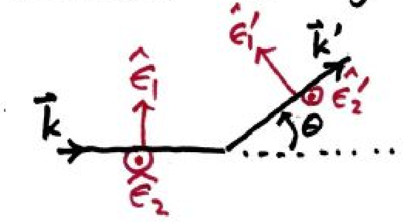
$$\sum_f = \sum_{\vec{p}\,'} \sum_{\vec{k}'} = \sum_{\vec{p}\,'} \frac{\Omega}{(2\pi)^3} k'^2 dk'\, d\Omega_{k'}$$

$$\delta(E_f - E_i) \xrightarrow[\text{low-energy limit}]{} \delta(\hbar c k' - \hbar c k) = \frac{1}{\hbar c}\delta(k'-k)$$

$$\frac{d\sigma}{d\Omega'} = \left(\frac{e^2}{mc^2}\right)^2 (\vec{\epsilon}\cdot\vec{\epsilon}\,')^2 = \alpha^2 \lambda\!\!\!^{-}_c{}^2 (\vec{\epsilon}\cdot\vec{\epsilon}\,')^2$$

$\alpha = e^2/\hbar c$ and $\lambda\!\!\!^{-}_c = \frac{\hbar}{mc}$.

Polarized scattering $\quad \frac{d\sigma}{d\Omega'} = \alpha^2 ƛ_c^2 (\vec{\epsilon}\cdot\vec{\epsilon}\,')^2$



$$\hat{\epsilon}_1 \cdot \hat{\epsilon}_1' = \cos\theta$$
$$\hat{\epsilon}_1 \cdot \hat{\epsilon}_2' = 0$$
$$\hat{\epsilon}_2 \cdot \hat{\epsilon}_1' = 0$$
$$\hat{e}_2 \cdot \hat{\epsilon}_2' = 1$$

$$\frac{d\sigma}{d\Omega'} = \alpha^2 ƛ_c^2 \{\cos^2\theta, 0, 0, 1\}$$

Unpolarized scattering
Average over $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$; sum over $\vec{e}_1'$ and $\vec{\epsilon}_2'$; $\Rightarrow$

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{unpol.}} = \frac{1}{2}\sum_{\vec{\epsilon}}\sum_{\vec{\epsilon}\,'} \alpha^2 ƛ_c^2 (\vec{\epsilon}\cdot\vec{\epsilon}\,')^2$$

$$(d\sigma/d\Omega_{\mathbf{k}'})_{\text{unpol.}} = \alpha^2\ ƛ_C^{\,2}\ (1 - ½ \sin^2\theta)$$

Thomson cross section

Examples of Thomson scattering (Wikipedia)

- The cosmic microwave background is linearly polarized as a result of Thomson scattering, as measured by DASI and more recent experiments.
- The solar K-corona is the result of the Thomson scattering of solar radiation from solar coronal electrons. NASA's STEREO mission generates three-dimensional images of the electron density around the sun by measuring this K-corona from two separate satellites.

inertial confinement

- In tokamaks, corona of ICF targets and other experimental fusion devices, the electron temperatures and densities in the plasma can be measured with high accuracy by detecting the effect of Thomson scattering of a high-intensity laser beam.
- Inverse-Compton scattering can be viewed as Thomson scattering in the rest frame of the relativistic particle.
- X-ray crystallography is based on Thomson scattering.

due Friday, January 29

Homework Problem 9
Calculate the total cross section for Thomson scattering, and express the result in millibarns (mb).

Homework Problem 10
“In fact, it can easily be demonstrated that it takes a photon emitted in the solar core many thousands of years to fight its way to the surface because of Thomson scattering.”
Prove it.