The book by Fetter and Walecka is concerned with the application of QFT to many-particle systems.

Relativistic QFT applies to something different:

- . cross sections for collisions
- . field interactions

For today’s lecture we’ll study an example related to particle interactions (but still using *nonrelativistic* QFT).

Then we’ll return to many-particle systems.

$-\infty\hbar\psi\hat{U}\rightarrow\rightarrow\rightarrow\rightarrow \int \delta$

Electron-electron scattering

- at nonrelativistic energies
- calculated using QFT

/1/ The Hamiltonian is $H = H_0 + H_1$.

The free Hamiltonian is

$$H_0 = \int \psi^\dagger \left(\frac{-\hbar^2}{2m}\right) \nabla^2 \psi(x)\, d^3x$$

Set h-bar = 1. At the end of the calculation we can restore the factors of h-bar by dimensional analysis.

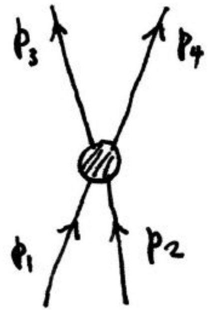
The interaction Hamiltonian is

$$H_I = \frac{1}{2}\int \psi^\dagger_\alpha(x)\, \psi^\dagger_\beta(y)\, V(x-y)\, \psi_\beta(y)\, \psi_\alpha(x)\, d^3x\, d^3y$$

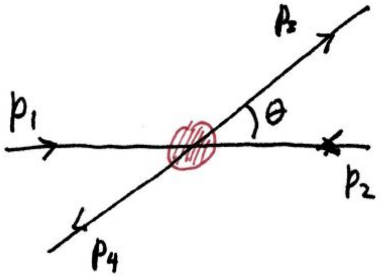
(spin indices, which I will suppress)

where $V(\mathbf{x}-\mathbf{y}) = e^2/|\mathbf{x}-\mathbf{y}|$

/2/ Kinematic variables; 4 momenta

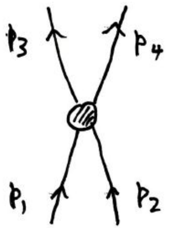


In the center of mass frame,



$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4$. We won't use this now.

/3/ The transition matrix element



← time $t \to +\infty$; state is $|\psi, t\rangle$

↑ time evolution with interactions

← time $t_0 \to -\infty$; state is $|e_1, e_2\rangle$

Start with the Schroedinger picture,

$$|\psi, t\rangle_S = e^{-iH(t-t_0)} \underbrace{|\psi, t_0\rangle_S}_{e^{-iH_0 t_0}|\psi, t_0\rangle_I} \qquad (\hbar = 1)$$

$$= e^{-iH_0 t} \underbrace{e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}}_{= \hat{U}(t, t_0)} |\psi, t_0\rangle_I$$

also, $|\psi, t\rangle_S = e^{-iH_0 t} |\psi, t\rangle_I$

$\therefore \quad |\psi, t\rangle_I = \hat{U}(t, t_0) |\psi, t_0\rangle_I$

Thus,

$$|\psi, t\rangle_I = \hat{U}(t, t_0)\,|\psi, t_0\rangle_I$$

Now remember, $t_0 \to -\infty$ and $t \to +\infty$.

So $\quad |\psi, t_0\rangle_I = |e_1, e_2\rangle$

If $H_I = 0$ then the interaction picture is the same as the Heisenberg picture.

The *transition probability amplitude* is

$$S = \langle e_3, e_4 | \psi, t\rangle_I$$

$$= \langle e_3, e_4 | \hat{U}(t, t_0) | e_1, e_2\rangle$$

Now recall,

$$\hat{U}(t, t_0) = T \exp\{ -i/\hbar \int_{t0}^{t} H_I(t')\, dt' \}$$

where $H_I(t')$ is H_I in the interaction picture; i.e., evolving according to H_0.

… the transition matrix element

Letting $t_0 \rightarrow -\infty$ and $t \rightarrow +\infty$; also, $\hbar = 1$;

$$S = \langle e_3 , e_4 | T \exp \{ - i \int_{-\infty}^{\infty} H_I(t)\, dt \} | e_1 , e_2 \rangle$$

where all states and operators are in the interaction picture.

Now apply perturbation theory.

<u>Zero-th order</u> $\hat{U}^{(0)} = 1$;

$S_{fi} = \langle f | \hat{U}^{(0)} | i \rangle = \delta_{fi}$; i.e., no scattering.

That does not contribute because we are interested in time evolution for which scattering ***does*** occur.

<u>First order</u> (or, <u>L</u>eading <u>O</u>rder)

$$\mathfrak{M} = \langle e_3 , e_4 | -i \int_{-\infty}^{\infty} H_I(t)\, dt \ | e_1 , e_2 \rangle$$

However, there will be some singular equations if we use $t \in (-\infty, \infty)$; so we'll make $t \in (-T , T)$ and later let $T \rightarrow \infty$.

$$H_I(t) = \tfrac{1}{2} \int d^3x\, d^3y\, V(\mathbf{x}-\mathbf{y}) \; \psi^\dagger(x)\psi^\dagger(y)\, \psi(y)\psi(x)$$

where $x = (\mathbf{x}, t)$ and $y = (\mathbf{y}, t)$.

(The instantaneous interaction b/c this is a nonrelativistic approximation.)

$\mathfrak{M} = < e_3 , e_4 | -i \int_{-T}^{T} H_I(t)\, dt \; | e_1 , e_2 >$

$H_I(t) = ½ \int d^3x\, d^3y\, V(\mathbf{x-y})$
$\psi^\dagger(x)\psi^\dagger(y)\, \psi(y)\psi(x)$

/5/ Calculation of the matrix element $\mathfrak{M}$

We could use Wick's theorem, but it'll be easier to go back to first principles.

$$\psi(x) = \sum_p \phi_p(\vec{x})\, e^{-i\varepsilon_p t}\, a_p \qquad (\hbar=1)$$

where $\varepsilon_p = \frac{\hbar^2 p^2}{2m} = \frac{p^2}{2m}$

and $\phi_p(\vec{x}) = \frac{1}{\sqrt{\Omega}} e^{i\vec{p}\cdot\vec{x}}\, u_s$ spin state $\binom{1}{0}$ or $\binom{0}{1}$

$p = (\vec{p}, s)$; $\sum_p = \sum_{\vec{p}} \sum_s$

$\psi(y)\,\psi(x)\; | e_1, e_2 >$

$= \psi(y)\, \psi(x)\, a_1^\dagger\, a_2^\dagger\, |0>$

$a_1 = a(\vec{p}_1, s_1)$

$\psi|0> = 0$

$= \{ \phi_{p_1}(\vec{x})\, e^{-i\varepsilon_1 t}\, \phi_{p_2}(\vec{y})\, e^{-i\varepsilon_2 t}$
$- \phi_{p_2}(\vec{x})\, e^{-i\varepsilon_2 t}\, \phi_{p_1}(\vec{y})\, e^{-i\varepsilon_1 t} \}\, |0>$

Note: $\{ \psi(x), a_1^\dagger \} = \phi_{p_1}(\vec{x})\, e^{-i\varepsilon_1 t}$

$< e_3, e_4 |\, \psi^\dagger(x)\, \psi^\dagger(y) = [\, \psi(y)\,\psi(x)\, | e_3, e_4 >]^\dagger$

$= <0|\, \{ \phi_{p_3}^\dagger(\vec{x})\, e^{i\varepsilon_3 t}\, \phi_{p_4}^\dagger(\vec{y})\, e^{i\varepsilon_4 t}$
$- \phi_{p_4}^\dagger(\vec{x})\, e^{i\varepsilon_4 t}\, \phi_{p_3}^\dagger(\vec{y})\, e^{i\varepsilon_3 t} \}$

$$\mathcal{M} = \frac{-i}{2} \int d^3x\, d^3y\, V(\vec{x}-\vec{y}) \int_{-T}^{T} \langle e_3 e_4 | \psi_x^{\dagger} \psi_y^{\dagger}\, \psi_y \psi_x | e_1 e_2 \rangle\, dt$$

$$\mathcal{M} = \frac{-i}{2} \int d^3x\, d^3y\, V(\vec{x}-\vec{y}) \int_{-T}^{T} e^{i(\varepsilon_3+\varepsilon_4-\varepsilon_1-\varepsilon_2)t}\, dt$$

$$\left[\phi^{\dagger}_{p_3}(\vec{x})\, \phi^{\dagger}_{p_4}(\vec{y}) - \phi^{\dagger}_{p_4}(\vec{x})\, \phi^{\dagger}_{p_3}(\vec{y}) \right]$$

$$\left[\phi_{p_1}(\vec{x})\, \phi_{p_2}(\vec{y}) - \phi_{p_2}(\vec{x})\, \phi_{p_1}(\vec{y}) \right]$$

c-number wave functions

The rest is just mathematics.
Remember, T→ ∞ and ħ = 1.

/4/ The time integral

$$\int_{-T}^{T} e^{i(\epsilon_f - \varepsilon_i)t}\, dt = \frac{2\sin[(\epsilon_f - \varepsilon_i)T]}{\varepsilon_f - \varepsilon_i}$$

$$\equiv I(\Delta\varepsilon, T)$$

where $\Delta\varepsilon = \varepsilon_f - \varepsilon_i$
$= \varepsilon_3 + \varepsilon_4 - \varepsilon_1 - \varepsilon_2$

Note: $\lim\limits_{T\to\infty} I(\Delta\varepsilon, T) = 2\pi\, \delta(\Delta\varepsilon)$;

that's conservation of energy $\Delta\varepsilon \cdot \Delta T \sim \hbar$

But we need to be careful, because we will need $|\mathcal{M}|^2$; $[\delta(x)]^2$ is undefined. So keep T finite for now.

$$I^2(\Delta\varepsilon, T) = \frac{4\sin^2(\Delta\varepsilon \cdot T)}{(\Delta\varepsilon)^2}$$

Theorem

$$\operatorname{limit}_{T\to\infty} \frac{1}{T} I^2(\Delta\varepsilon, T) = 4\pi\, \delta(\Delta\varepsilon)$$

We'll need this later.

Homework problem

/5/ *Finish calculating the transition matrix element*

$$\mathcal{M} = \frac{-i}{2}\int d^3x\, d^3y\, V(\vec{x}-\vec{y})\quad I(\Delta\varepsilon, T)$$

$$\left[\phi^+_{p_3}(\vec{x})\,\phi^+_{p_4}(\vec{y}) - \phi^+_{p_4}(\vec{x})\,\phi^+_{p_3}(\vec{y})\right] \leftarrow \text{c-number}$$

$$\left[\phi_{p_1}(\vec{x})\,\phi_{p_2}(\vec{y}) - \phi_{p_2}(\vec{x})\,\phi_{p_1}(\vec{y})\right] \swarrow \text{wave functions}$$

First trick is to replace

$$\phi^+_{p_3}(x)\,\phi^+_{p_4}(y) - \phi^+_{p_4}(x)\,\phi^+_{p_3}(y)$$

by $\phi^+_{p_3}(x)\,\phi^+_{p_4}(y) \times 2$.

That's because of exchange symmetry:

$$V(x-y)\left[-\phi^+_4(x)\,\phi^+_3(y)\right]\left[\phi_1(x)\phi_2(y) - \phi_2(x)\phi_1(y)\right]$$

$\downarrow\quad x \leftrightarrow y$ and exchange spin indices

$$V(y-x)\left[-\phi^+_4(y)\,\phi^+_3(x)\right]\left[\phi_1(y)\phi_2(x) - \phi_2(y)\phi_1(x)\right]$$

$$= V(x-y)\;\phi^+_3(x)\,\phi^+_4(y)\left[\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x)\right]$$

Second trick is to write

$$V(\vec{x}-\vec{y}) = \frac{e^2}{|\vec{x}-\vec{y}|} = \int \frac{d^3q}{(2\pi)^3}\, \frac{e^{i\vec{q}\cdot(\vec{x}-\vec{y})}}{q^2}\, 4\pi e^2$$

That's because

$$-\nabla^2\left(\frac{e^2}{r}\right) = 4\pi\,\delta^3(\vec{r})\quad e^2$$

Third trick

$$\int d^3x\; e^{i\vec{q}\cdot\vec{x}}\, e^{-i\vec{p}_3\cdot\vec{x}}\, e^{i\vec{p}_1\cdot\vec{x}}$$

$$= \Omega\,\delta_{Kr}(\vec{q}\,;\vec{p}_3-\vec{p}_1)$$ periodic boundary conditions

So

$$\mathcal{M} = \frac{-i}{2}\times 2\times 4\pi e^2 \times I \times \Omega^2\,\frac{1}{(\sqrt{\Omega})^4}$$

$$\int\frac{d^3q}{(2\pi)^3}\left\{\delta_{Kr}(\vec{q}\,;\vec{p}_3-\vec{p}_1)\,\delta_{Kr}(-\vec{q}\,;\vec{p}_4-\vec{p}_2) - \delta_{Kr}(\vec{q}\,;\vec{p}_3-\vec{p}_2)\,\delta_{Kr}(-\vec{q}\,;\vec{p}_4-\vec{p}_1)\right\}\frac{1}{q^2}$$

$$\mathcal{M} = \frac{-i}{2}\times 2 \times 4\pi e^2 \times I \times \Omega^2 \frac{1}{(\sqrt{\Omega})^4}$$

$$\underbrace{\int \frac{d^3q}{(2\pi)^3}} \Big\{ \delta_{Kr}(\vec{q}\,; \vec{p}_3-\vec{p}_1)\,\delta_{Kr}(-\vec{q}\,; \vec{p}_4-\vec{p}_2) - \delta_{Kr}(\vec{q}\,; \vec{p}_3-\vec{p}_2)\,\delta_{Kr}(-\vec{q}\,; \vec{p}_4-\vec{p}_1) \Big\} \frac{1}{q^2}$$

in a finite volume this should be $\frac{1}{\Omega}\sum_{\vec{q}}$ b/c $\vec{q} = \frac{3\pi}{L}\vec{n}$

$$\delta^3\vec{q} = \frac{(2\pi)^3}{\Omega}\delta^3 n$$

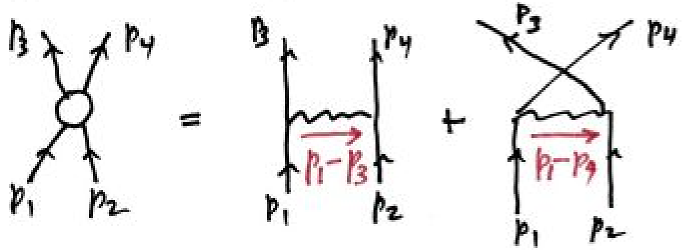
$$\mathcal{M} = \frac{-i}{\Omega} I(\Delta z, T)\, \delta_{Kr}(\vec{p}_3+\vec{p}_4\,; \vec{p}_1+\vec{p}_2)$$

$$\left\{ \frac{4\pi e^2}{(\vec{p}_1-\vec{p}_3)^2} A - \frac{4\pi e^2}{(p_1 - p_4)^2} B \right\}$$

$$A = u_3^{\dagger} u_1 \; u_4^{\dagger} u_2 = \delta(s_3, s_1)\,\delta(s_4, s_2)$$

$$B = u_3^{\dagger} u_2 \; u_4^{\dagger} u_1 = \delta(s_3, s_2)\,\delta(s_4, s_1)$$

In relativistic QFT this comes from 2 Feynman diagrams :



Next : Calculate the scattering cross section .

Homework Problem due Friday February 19

Problem 22.
Prove

$$\lim_{T \to \infty} I^2(\Delta\varepsilon, T)/T = 4\pi\, \delta(\Delta\varepsilon)$$

where

$$I(\Delta\varepsilon, T) = \int_{-T}^{T} \exp[i(\Delta\varepsilon)t]\; dt$$