THE NEARLY DEGENERATE IDEAL BOSE-EINSTEIN GAS

Review some results from the last lecture…

First we assumed that there is no Bose-Einstein (BE) condensate at temperature T.
Then …

$$N(\vec{k}) = \frac{1}{C e^{E(k)/kT} - 1} \quad \text{(bosons)}$$

where $E(k) = \hbar^2 k^2 /(2m)$;

and

$$N_{tot} = \sum_{\vec{k}} N(\vec{k}) = \frac{\Omega}{(2\pi)^3} \int \frac{d^3k}{C e^{E/kT} - 1}$$

But then we found that there is no solution for $T < T_C$ where

$$\nu = \frac{\sqrt{2}}{2\pi^2} \left(\frac{m k T_c}{\hbar^2} \right)^{3/2} f(0)$$

So for $T < T_C$ there must be a significant fraction of particles in the ground state (i.e., $\mathbf{k} = 0$);

$$N_{tot} = N(0) + \frac{\Omega}{(2\pi)^3} \int \frac{d^3k}{e^{E/kT} - 1}$$

$$\downarrow$$

$$N(0) = F \, N_{tot}$$

That is the IDEAL Bose-Einstein gas. Now include the effects of H_I , to estimate the *excitation energies.*

Consider very low T ; i.e., $T \ll T_C$

Remember this: If many particles are in the same quantum state then the field behaves *almost classically*.

We have $[b_0, b_0^\dagger] = b_0 b_0^\dagger - b_0^\dagger b_0 = 1$; but this is << the number operator $b_0^\dagger b_0$, which is of order N_{tot} .

So we can approximate

$$b_0^\dagger b_0 \approx N \quad \text{and} \quad b_0 \approx \overline{b_0^\dagger} \approx \sqrt{N} \; ;$$

in other words, we can approximate these operators by c-numbers.

And what about the operators b_k and $b_k^\dagger$ with $k \neq 0$?

As T approaches 0, we can neglect higher order products of b_k and $b_k^\dagger$.

We start again with

$$H = \sum_k \hbar^2 k^2/(2m) \; b_k^\dagger b_k$$

$$+ (1/\Omega) \sum_{k1, k2, q} v(\mathbf{q}) \; b_{\mathbf{k1+q}}^\dagger \; b_{\mathbf{k2-q}}^\dagger \; b_{\mathbf{k2}} \; b_{\mathbf{k1}}$$

and make some approximations.

- $b_0 \approx b^\dagger \approx \sqrt{N}$;
- neglect terms cubic or quartic in $b_\mathbf{k}$ and $b_\mathbf{k}^\dagger$ for $\mathbf{k} \neq 0$;
- $b_0^{\;2} + \sum_k b_k^\dagger \, b_\mathbf{k} = N$; so $b_0^{\;4} = N^2 - 2N \sum b^\dagger b$
- for simplicity, write $v(\mathbf{q}) = v_0$ (constant) .

$$V \approx N^2 v_0/\Omega$$

$$+ N v_0/\Omega \sum_q (b_\mathbf{q}^\dagger \, b_{-\mathbf{q}}^\dagger + b_\mathbf{q} \, b_{-\mathbf{q}} + 2 \, b^\dagger \, b_\mathbf{q})$$

$n = N/\Omega$

The canonical transformation

(Bogoliubov, 1947)

Let L_k be a c-number, to be determined. Define

$$M = \sqrt{1 - L_k^2}$$

$$a_\mathbf{k} = (b_\mathbf{k} + L_k b_{-\mathbf{k}}^\dagger) /M$$

$$a_\mathbf{k}^\dagger = (b_\mathbf{k}^\dagger + L_k b_{-\mathbf{k}}) /M$$

Note that the commutation relations are invariant,

$$[a_\mathbf{k} , a_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} \quad ;$$

$$[a_\mathbf{k} , a_{\mathbf{k}'}] = 0 \text{ and } [a_\mathbf{k}^\dagger , a_\mathbf{k}^\dagger] = 0 \quad ;$$

i.e., the same as for the $b_\mathbf{k}$ and $b_\mathbf{k}^\dagger$.

The inverse transformation is

$$b_\mathbf{k} = (a_\mathbf{k} - L_k a_{-\mathbf{k}}^\dagger) /M$$

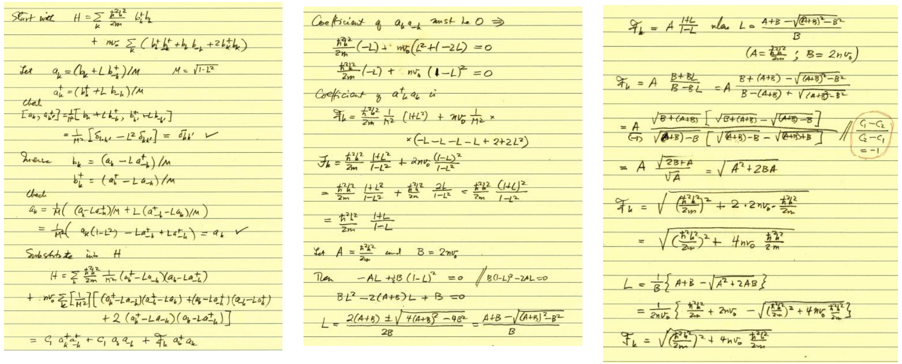
$$b_\mathbf{k}^\dagger = (a_\mathbf{k}^\dagger - L_k a_{-\mathbf{k}}) /M$$

Now rewrite the Hamiltonian in terms of $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$.

$$H = \sum_k \hbar^2 k^2 /(2m) \; b_\mathbf{k}^\dagger b_\mathbf{k} + Nnv_0$$
$$+ nv_0 \sum_k (b_\mathbf{k}^\dagger b_{-\mathbf{k}}^\dagger + b_\mathbf{k} b_{-\mathbf{k}} + 2\, b_\mathbf{k}^\dagger b_\mathbf{k})$$

There will be terms proportional to $a_\mathbf{k} a_{-\mathbf{k}}$ and $a_\mathbf{k}^\dagger a_{-\mathbf{k}}^\dagger$. Make their coefficients 0 by a suitable choice of L_k. Then the remaining terms will be proportional to $a_\mathbf{k}^\dagger a_\mathbf{k}$ …

After several pages of algebra,



the result is ___

$$L_k = \frac{1}{2nv_0}\left\{ \frac{\hbar^2 k^2}{2m} + 2nv_0 - \mathcal{E}(k) \right\}$$

where

$$\mathcal{E}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m}\right)^2 + 4nv_0 \frac{\hbar^2 k^2}{2m}}$$

then

$$H = Nnv_0 + \sum_k \mathcal{E}(k)\, a_k^\dagger a_k$$

Quasi-particles

The final hamiltonian (H) describes a theory of *noninteracting "particles" with single "particle" energies = $\varepsilon(\mathbf{k})$.*

Quasi-particles

- For long wavelengths,

  ($k = 2\pi / \lambda$; ∴ long wavelengths ⇒ small k)

  $\varepsilon(k) \approx c_s\ \hbar k$ (units?)

  $c_s = \sqrt{2\, nv_0 / m}$
  = the speed of sound

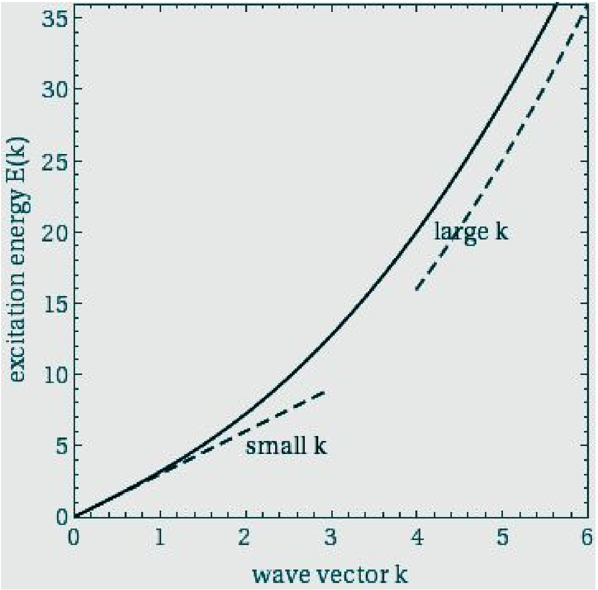
These quantized waves are called ***phonons*** (a "*collective motion*") .

- For short wavelengths, (large k)

  $\varepsilon(k) \approx \hbar^2 k^2 / 2m$

so these approximate single atoms w/ ***w.v.*** = k

Excitations of the BE condensate

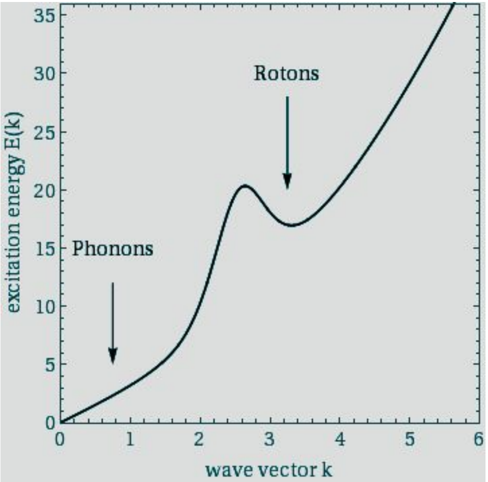


Superfluidity of He-4

Consider a large object moving slowly through the superfluid. Although there is no true energy gap, excitation of phonons is negligible;
the density of phonon states is small down to k = 0.

<u>Superfluid He-4</u>

Landau proposed that the spectrum of excitations in the superfluid phase of He-4 looks like this:



Landau and Lifschitz (19??);
Onsager (1947);
Feynman (1955)

Superfluid helium-4
From Wikipedia, the free encyclopedia.

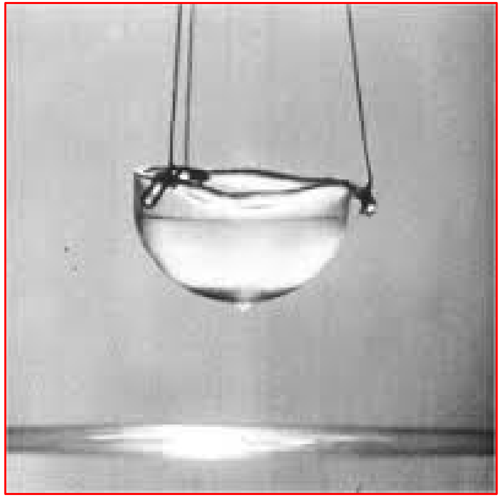
A superfluid is a state of matter in which the matter behaves like a fluid with zero viscosity. The substance, which looks like a normal liquid, flows without friction past any surface, which allows it to continue to circulate over obstructions and through pores in containers which hold it, subject only to its own inertia.

L. D. Landau's phenomenological and semi-microscopic theory of superfluidity of helium-4 earned him the Nobel Prize in physics, in 1962. Assuming that sound waves are the most important excitations in helium-4 at low temperatures, he showed that helium-4 flowing past a wall would not spontaneously create excitations if the flow velocity was less than the sound velocity. In this model, the sound velocity is the "critical velocity" above which superfluidity is destroyed.

Landau thought that vorticity entered superfluid helium-4 by vortex sheets, but such sheets have since been shown to be unstable. Lars Onsager and, later independently, Feynman showed that vorticity enters by quantized vortex lines. They also developed the idea of quantum vortex rings. Hendrik van der Bijl in the 1940s,[23] and Richard Feynman around 1955,[24] developed microscopic theories for the roton, which was shortly observed with inelastic neutron experiments by Palevsky. Later on, Feynman admitted that his model gives only qualitative agreement with experiment.[25][26]

I, Alfred Leitner, took this photograph as part of my movie "Liquid Helium, Superfluid" - Own work (1962)

The liquid helium is in the superfluid phase. A thin invisible film creeps up the inside wall of the cup and down on the outside. A drop forms. It will fall off into the liquid helium below. This will repeat until the cup is empty - provided the liquid remains superfluid.



Homework Problem due Friday, February 26

Problem 26.
Read this paper:

Bewley, G. P., Lathrop, D. P. and Sreenivasan, K. R. (2006). Visualization of quantized vortices. Nature. 441, 588.

In one paragraph (*written in your own words*) with one figure (*drawn by you*), summarize the paper.