Harris, Chapter 9

Quasi particles in plasmas and metals

Phonons and Plasmons

The system consists of *MANY* charged particles—electrons and ions—with total charge = 0.

1/ The particles interact with an electrostatic potential, $\Phi_1(\mathbf{x},t)$;

$$H = \sum_s \int \Psi_s^\dagger \left[-\hbar^2\nabla^2/2m + e_s\,\Phi_1\right] \Psi_s \; d^3x$$

(s = species; s = *e* and *i*)

2/ We'll solve that QFT (approximately) in terms of a dielectric function.

3/ The particles create an electrostatic field, $\Phi_2(\mathbf{x},t)$, which we'll calculate.

4/ And finally use the *self-consistent field approximation*,

$$\Phi_1(\mathbf{x},t) = \Phi_2(\mathbf{x},t)$$

Start by expanding $\Psi_s(\mathbf{x})$ in plane wave states,

$$\Psi_s(x) = \sum_{\mathbf{k}} b_{\mathbf{sk}} \exp(i\, \mathbf{k} \cdot \mathbf{x}) \;/\sqrt{\Omega}$$

The free particle Hamiltonians become

$$H_{s0} = \sum_{\mathbf{k}} (\hbar^2 k^2 /2m)\; b_{\mathbf{sk}}^{\dagger} b_{\mathbf{sk}} \qquad (s{=}e \text{ or } i)$$

The interaction Hamiltonians become

$$H_{s1} = (1/\Omega) \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \; e_s\, \varphi_1(\mathbf{k}_2 - \mathbf{k}_1)\, b_{s,\mathbf{k1}}^{\dagger} b_{s,\mathbf{k2}}$$

Here $\varphi_1(\mathbf{q})$ is the Fourier transform of $\Phi_1(\mathbf{x})$,

$$\varphi_1(\mathbf{q}) = \int \Phi_1(\mathbf{x}) \exp(i\, \mathbf{q} \cdot \mathbf{x})\, d^3x$$

We'll also need $\Phi_1(\mathbf{x}) = 1/\Omega \sum_{\mathbf{q}} \varphi_1(-\mathbf{q}) \exp(i\, \mathbf{q} \cdot \mathbf{x})$.

Now consider the operator $b_{\mathbf{sk'}}^{\dagger} b_{\mathbf{sk}}$,

which annihilates a particle with momentum $\hbar\mathbf{k}$ and creates a particle with momentum $\hbar\mathbf{k}'$.
(This is part of a scattering process.)

In the Heisenberg picture,

$$i\hbar\; \partial/\partial t\; b_{\mathbf{sk'}}^{\dagger} b_{\mathbf{sk}} = - [\, H_{s0} + H_{sI} ,\, b_{\mathbf{sk'}}^{\dagger} b_{\mathbf{sk}} \,]$$

$$= (E_{sk} - E_{sk'})\, b_{sk'}^{\dagger} b_{sk} - \frac{e_s}{\Omega} \sum_p \Big[\varphi_1(p-k)\, b_{\mathbf{sk'}}^{\dagger} b_{\mathbf{sp}} - \varphi_1(k'-p)\, b_{\mathbf{sp}}^{\dagger} b_{\mathbf{sk}} \Big]$$

Exercise. Derive the time-evolution equation, for both fermions and bosons.

A DISTRIBUTION FUNCTION

Define

$$F_s(\mathbf{k'}, \mathbf{k}\,; t) = \sum P(\alpha) < \alpha |\; b_{\mathbf{sk'}}^{\dagger} b_{\mathbf{sk}} \;| \alpha >$$

("density matrix"; $|\alpha>$'s are states of the system and $P(\alpha)$ = the probability of state $|\alpha>$)

The time evolution

$$i\hbar \;\partial/\partial t \;\; F_s(\mathbf{k'},\mathbf{k}\,; t) =$$

$$= (E_{sk} - E_{sk'}) \; F_s(\mathbf{k'},\mathbf{k}\,; t)$$

$$- \frac{e_s}{\Omega} \sum_p \Big[\varphi_1(p-k) F_s(k', p\,; t) - \varphi_1(k'-p) F_s(p k\,; t) \Big]$$

Note how this D.F. is related to the mean number density; recall,

$$<n_s(\mathbf{x}\,, t)> = \sum P(\alpha) < \alpha |\; \Psi_s^{\dagger}(\mathbf{x},t)\Psi_s(\mathbf{x},t) \,| \alpha >$$

$$= \sum_{\mathbf{k'}} \sum_{\mathbf{k}} F_s(\mathbf{k'},\mathbf{k}\,; t)\, e^{i(k-k').x} / \Omega$$

If we know the mean charge density, then we can calculate the corresponding electrostatic potential from Poisson's equation;

$$-\nabla^2 \Phi_2 = \sum_s \; 4\pi\, e_s < n_s(x,t) >$$

(gaussian EM units)

THE APPROXIMATE DISTRIBUTION FUNCTION

We need to make another approximation.

The total charge is zero. In the 0^{th} approximation, the charge density (summed over s) is 0 and the potential Φ_2 is 0.

In the 0^{th} approximation, the distribution function for the electrons is the Fermi-Dirac distribution. We'll also take the ion distribution to be a Fermi-Dirac distribution.

(Essentially equivalent to "jellium".)

In the *1^{st} approximation*, treat δF_s and Φ_2 as small quantities.

(an example of first order perturbation theory; *neglect products of small quantities)*

We are looking for harmonic waves, so we'll try this ansatz,

$$F_s(\mathbf{k'}, \mathbf{k}\,;t) = F_{s0}(\mathbf{k})\delta_{\mathbf{k'},\mathbf{k}} + F_{s1}(\mathbf{k'},\mathbf{k})\, e^{-i\omega t}$$

where $F_{s0}(\mathbf{k})$ is the Fermi-Dirac distribution.

In terms of *velocities*,

$F_{s0}(\mathbf{k}) = f_{s0}(\mathbf{v})$ where $\mathbf{v} = \hbar\mathbf{k}/m$

$$f_{s0}(\mathbf{v}) = \begin{cases} \dfrac{n}{\frac{4}{3}\pi v_{Fs}^3} & \text{for } v < v_{Fs} \\ 0 & \text{for } v > v_{Fs} \end{cases} \quad ; \quad v_{Fs} = \frac{\hbar}{m_s}\left(\frac{3n}{4\pi}\right)^{1/3}$$

Plug in the ansatz into the equation for $F_s(\mathbf{k'},\mathbf{k}\,;t)$

$$i\hbar \frac{\partial}{\partial t}\left[F_{s0}(k)\,\delta_{k'k} + F_{s1}(k'k)\,e^{-i\omega t}\right]$$

$$= (E_{sk} - E_{sk'})\left[F_{s0}(k)\,\delta_{k'k} + F_{s1}(k'k)\,e^{-i\omega t}\right]$$

$$- \frac{e_s}{\Omega}\sum_p \left[\varphi_1(p-k)\,F_{s0}(k')\,\delta_{k'p} - \varphi_1(k'-p)\,F_{s0}(k)\,\delta_{kp}\right]$$

neglecting terms $\propto \varphi_1 F_{1s}$.

$$\left[\hbar\omega - (E_{sk}-E_{sk'})\right] F_{s1}(k'k)$$

$$= -\frac{e_s}{\Omega}\,\underbrace{\varphi_1(k'-k)\,e^{i\omega t}}_{\varphi_1(k'-k\,;\,t)\,e^{i\omega t}}\left[F_{s0}(k') - F_{s0}(k)\right]$$

$$F_{1s}(k'k;t) = -\frac{e_s}{\Omega}\,\frac{\bar{\varphi}_1(k'-k)\left[F_{s0}(k') - F_{s0}(k)\right]}{\hbar\omega - (E_{sk} - E_{sk'})}$$

where

$$\varphi_1(k'-k;t) = \bar{\varphi}_1(k'-k)\,e^{-i\omega t}$$

The self consistent mean field

Poisson's equation,

$$-\nabla^2\Phi_2 = \sum_s 4\pi\, e_s < n_s(x,t) >$$

$$= \sum_s 4\pi e_s \sum_{kk'} \left[F_{s0}(k)\,\delta_{k'k} + F_{s1}(k'k) e^{-i\omega t} \right] \frac{e^{i\vec{q}\cdot\vec{x}}}{\Omega}$$

$\uparrow$ zero $\qquad \vec{q} \equiv \vec{k} - \vec{k}'$

$$= \sum_{skk'} \frac{4\pi e_s}{\Omega} F_{s1}(k'k)\, e^{-i\omega t}\, e^{i\vec{q}\cdot\vec{x}}$$

Thus we can write

$$\Phi_2(\mathbf{x},t) = (1/\Omega) \sum_{\mathbf{q}} \overline{\boldsymbol{\varphi}_2}(-\boldsymbol{q})\ e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\ e^{-i\omega t}\ ;$$

and we have

$$\frac{1}{\Omega} q^2 \bar{\varphi}_2(-\vec{q}) = \sum_s \frac{4\pi e_s}{\Omega} \sum_{\vec{k}} F_{s1}(\vec{k}-\vec{q}, \vec{k})$$

$$\frac{1}{\Omega} q^2 \bar{\varphi}_2(-\vec{q}) = \sum_s \frac{4\pi e_s}{\Omega} \sum_{\vec{k}} \frac{e_s}{\Omega} \bar{\varphi}_1(-q) \frac{F_{s0}(k) - F_{s0}(k-q)}{\hbar\omega - (E_{sk} - E_{sk'})}$$

Finally, we make the self-consistent field approximation: $\Phi_2(\mathbf{x},t) = \Phi_2(\mathbf{x},t)$; or, $\boldsymbol{\varphi}_2(-\mathbf{q}) = \boldsymbol{\varphi}_1(-\mathbf{q}) = \boldsymbol{\varphi}(-\mathbf{q})$

$$q^2 \bar{\varphi}(-\vec{q}) = \sum_s \frac{4\pi e_s^2}{} \int \frac{d^3k}{(2\pi)^3} \frac{F_{s0}(k) - F_{s0}(k-q)}{\hbar\omega - (E_{sk} - E_{sk'})} \bar{\varphi}(-q)$$

<u>The dielectric function of the plasma is given by Equation (9.20).</u>

$$\varepsilon(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{q^2} \int \frac{d^3k}{(2\pi)^3} \frac{F_{s0}(k-q) - F_{s0}(k)}{\hbar\omega - (E_{sk} - E_{sk'})}$$

I might have a sign error!

The self-consistent mean field theory is simply

$$\varepsilon(\mathbf{q},\omega)\, \boldsymbol{\varphi}(-\mathbf{q}) = 0 \quad .$$

But there is a problem!
The denominator might be 0, giving an improper integral.
Landau: replace ω by $\omega+i\eta$ where $\eta \to 0+$.
[OK, but why not $\omega-i\eta$?]

The Plemelj Formulas

An important mathematical identity, involving generalized functions;

$$\frac{1}{x \pm i\eta} \xrightarrow[\eta \to 0+]{} P\frac{1}{x} \mp i\pi\,\delta(x)$$

"principal value"

The generalized functions on the RHS are defined w.r.t. integration;

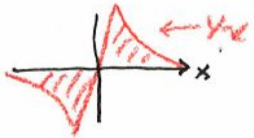
$$\int_{-\infty}^{\infty} \delta(x)\,\tau(x)\,dx = \tau(0)$$

$$\int_{-\infty}^{\infty} P\frac{1}{x}\,\tau(x)\,dx = \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{\tau(x)}{x}\,dx$$

$(\epsilon \to 0)$

Proof

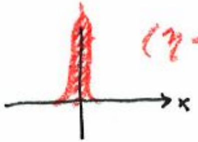
$$P\left(\frac{1}{x}\right) = \frac{1}{2}\left[\frac{1}{x+i\eta} + \frac{1}{x-i\eta}\right] = \frac{x}{x^2+\eta^2}$$



$(\eta \to 0+)$

$$\delta(x) = \frac{-1}{2\pi i}\left[\frac{1}{x+i\eta} - \frac{1}{x-i\eta}\right] = \frac{-1}{2\pi i}\left(\frac{-2i\eta}{x^2+\eta^2}\right)$$

$$= \frac{1}{\pi}\,\frac{\eta}{x^2+\eta^2}$$



$(\eta \to 0^+)$

Homework Problem.

Derive the time evolution equation for the distribution function

$$F_s(\,\mathbf{k}',\mathbf{k}\,;\,t\,)$$

that was defined in the lecture of February 19.