Chapter 9 : Quasi particles in plasmas and metals

Phonons and Plasmons

… continued

Recall the result derived last time:
The self-consistent field theory with
$\Phi(\mathbf{x},t) = 1/\Omega \sum_{\mathbf{q}} \varphi(-\mathbf{q}) \exp(i\, \mathbf{q}.\mathbf{x} - i\, \omega t)$
must have

$$\varphi(-\mathbf{q})\, \varepsilon(\mathbf{q},\omega) = 0$$

where the "dielectric function" is

$$\varepsilon(\mathbf{q},\omega) = 1 + \sum_{s}\sum_{k} \frac{4\pi e_s^{\,2}}{\Omega\, q^2} \; \frac{F_{s0}(\mathbf{k}) - F_{s0}(\mathbf{k} - \mathbf{q})}{\hbar\omega - \hbar\omega_s(\mathbf{k}, \mathbf{k} - \mathbf{q})}$$

(Harris Eq.9-20)

The system consists of MANY charged particles—electrons and ions—with total charge = 0.

1/ The particles interact with an electrostatic potential, $\phi_1(\boldsymbol{x},t)$.

$$H = \sum_{s} \int \Psi^{*} [-\hbar^2 \nabla^2/2m + e_s \Phi_1]\, \Psi \; d^3x$$

(s = species; s = e and i)

2/ We'll solve that QFT (approximately) in terms of a dielectric function.

3/ The particles create an electrostatic field, $\Phi_2(\boldsymbol{x},t)$.

4/ The self-consistent field approximation:
$\Phi_1(\boldsymbol{x},t) = \Phi_2(\boldsymbol{x},t)$

$$\varepsilon(\mathbf{q},\omega) = 1 + \sum_{s}\sum_{\mathbf{k}} \frac{4\pi\, e_s^{\,2}}{\Omega\, q^2} \; \frac{F_{s0}(\mathbf{k}) - F_{s0}(\mathbf{k-q})}{\hbar\omega - \hbar\omega_s\,(\mathbf{k}, \mathbf{k-q})}$$

Now we will interpret the result, in terms of ***quasi-particles***.

Waves can exist within the material provided that $\varepsilon(\mathbf{q},\omega) = 0$. This equation implies a "dispersion relation",

$$\omega = \omega(\mathbf{q}).$$

The excitations of the plasma are like field quanta, which we call quasi-particles.

$$\varepsilon(\mathbf{q},\omega) = 1 + \sum_{s}\sum_{\mathbf{k}} \frac{4\pi\, e_s^{\,2}}{\Omega\, q^2} \; \frac{F_{s0}(\mathbf{k}) - F_{s0}(\mathbf{k-q})}{\hbar\omega - \hbar\omega_s\,(\mathbf{k}, \mathbf{k-q})}$$

Rewrite this in terms of the *velocity* distribution;

$$\hbar\,\mathbf{k} = m_s\,\mathbf{v}\,; \qquad d^3k = (m_s/\hbar)^3\, d^3v\;;$$

$$F_{s0}(\mathbf{k})\, d^3k/(2\pi)^3 = f_{s0}(\mathbf{v})\, d^3v$$

Also, the denominator =

$$\hbar\,\omega - [\, E_s(\mathbf{k}) - E_s(\mathbf{k-q})\,]$$
$$= \hbar\,\omega - \hbar^2\mathbf{k}^2/2m + \hbar^2(\mathbf{k-q})^2/2m$$
$$= \hbar\;[\,\omega - \mathbf{v.q} + \hbar q^2/2m_s\,]$$

So,

$$\varepsilon(\mathbf{q},\omega) = 1 + \sum_s \frac{4\pi e_s^{\,2}}{\hbar q^2} \int d^3v \, \frac{f_{s0}(\mathbf{v}) - f_{s0}(\mathbf{v} - \hbar\mathbf{q}/m_s)}{\omega - \mathbf{v}.\mathbf{q} + \hbar q^2/2m_s}$$

(Eq. 9-23)

A CLASSICAL APPROXIMATION,

not really necessary,

but just to simplify the analysis.

Take the limit $\hbar \longrightarrow 0$ in the integrand.

$\therefore$ numerator $\longrightarrow$ $-(\partial f_{s0}/\partial\mathbf{v})\cdot(-\hbar\mathbf{q}/m_s)$

(This is not really *classical physics*, because $f_{s0}(\mathbf{v})$ = the Fermi-Dirac distribution; i.e., the Pauli exclusion principle is included.)

$$\varepsilon(\mathbf{q},\omega) = 1 + \sum_s \frac{4\pi e_s^{\,2}}{m_s q^2} \int d^3v \, \frac{\mathbf{q}\cdot\partial f_{s0}/\partial\mathbf{v}}{\omega - \mathbf{v}.\mathbf{q}}$$

THE IMPROPER INTEGRAL

□ Landau:
we should replace ω by $\omega + i\,\eta$, with $\eta \longrightarrow 0^+$.

□ Plemelj formula:
$1/(x+i\eta) = P\,(1/x) - i\pi\,\delta(x)$
so now $\varepsilon(\mathbf{q},\omega)$ may be complex.

□ $\varepsilon(\mathbf{q},\omega) = 0$:
so ω may be complex.

□ If $\mathfrak{Im}(\omega)$ is negative then the waves are damped.

□ We won't use the Plemelj formula.

$$\varepsilon(\mathbf{q},\omega) = 1 + \sum_s \frac{4\pi e_s^{\,2}}{m_s q^2} \int d^3v \, \frac{\mathbf{q}\cdot\partial f_{s0}/\partial\mathbf{v}}{\omega - \mathbf{v}.\mathbf{q}}$$

THE FERMI-DIRAC DISTRIBUTION FUNCTION

Even at room temperature we can treat the conduction electrons as a degenerate gas of fermions. Then,

$$f_{s0}(v) = \begin{cases} n\,/\,(4/3\,\pi\, v_{Fs}^{\;3}) & \text{for } v < v_{Fs} \\ 0 & \text{for } v > v_{Fs} \end{cases}$$

where the Fermi velocity is

$v_{Fs} = (\hbar/m_s)\,(3\pi^2 n)^{1/3}$.

$$\frac{\partial f_{s0}}{\partial \boldsymbol{v}} = \frac{3n}{4\,\pi\, v_{Fs}^{\;3}}\, \frac{\boldsymbol{v}}{v}\, \{-\delta(v_{Fs} - v)\}$$

THE VELOCITY INTEGRAL

$$\varepsilon_c(\vec{q},\omega) = 1 + \sum_s \frac{4\pi e_s^2}{m_s q^2} \int d^3v\, \frac{\vec{q}\cdot\partial f_{s0}/\partial\vec{v}}{\omega - \vec{v}\cdot\vec{q} + i\eta} \qquad (9.24)$$

$$= 1 - \sum_s \frac{4\pi e_s^2}{m_s q^2}\, \frac{3n}{4\pi v_s^3 v_s} \int d^3v\, \frac{\vec{q}\cdot\vec{v}\;\delta(v_{Fs}-v)}{\omega - \vec{q}\cdot\vec{v}}$$

$$\int \frac{2\pi\, v^2\, dv\, d(\cos\theta)}{\omega - qv\cos\theta}\, qv\cos\theta\, \delta(v_{Fs}-v)$$

$$= 2\pi q\, v_{Fs}^3 \int_{-1}^{1} \frac{x\,dx}{\omega - q v_{Fs} x}$$

$$= \frac{2\pi q\, v_{Fs}^3}{q v_{Fs}} \int_{-1}^{1} \frac{x\,dx}{z_s - x} \qquad \boxed{z_s \equiv \frac{\omega}{q v_{Fs}}}$$

$$= 2\pi v_{Fs}^2 \left[-2 + z_s \ln\frac{z_s+1}{z_s-1}\right]$$

$$\varepsilon_c(\vec{q},\omega) = 1 + \sum_s \frac{6\pi e_s^2 n}{m_s q^2 v_{Fs}^2}\left[2 - z_s \ln\frac{z_s+1}{z_s-1}\right]$$

Recall $v_{Fs} = \frac{\hbar}{m_s}(3\pi^2 n)^{1/3}$

$$\varepsilon_c(\vec{q},\omega) = 1 + \sum_s \frac{K_s^2}{q^2}\left[2 - z_s \ln\frac{z_s+1}{z_s-1}\right]$$

where $K_s^2 = \left(\frac{24 n}{\pi}\right)^{1/3} \frac{e_s^2 m_s}{\hbar^2}$

WE SEEK THE SOLUTIONS OF

$$\varepsilon(\mathbf{q},\omega) = 0$$

In other words we want to know ω as a function of **q**,

$$\omega = \omega(q) \quad \text{“dispersion relation”}$$

q is real; ω might be complex;
if ω has a negative real part then the waves are damped in time.

Let's put in some real parameter values.

Metallic sodium.

$e_e = -e$ and $m_e = 9.11 \times 10^{-31}$ kg

$e_i = e$ and $m_i = 23 \times 1.66 \times 10^{-27}$ kg

$$n = \frac{\rho}{m_i} = \frac{970 \text{ kg/m}^3}{m_i} = 2.54 \times 10^{28} \text{ m}^{-3}$$

$$K_e^2 = \left(\frac{24n}{\pi}\right)^{1/3} \frac{e^2 m_e}{\hbar^2} = \left(1.046 \times 10^{10} \text{ m}^{-1}\right)^2$$

$$K_i^2 = K_e^2 \frac{m_i}{m_e} \quad (\text{note: } K_i^2 \gg K_e^2)$$

$$v_{Fe} = \frac{\hbar}{m_e}\left(3\pi^2 n\right)^{1/3} = 1.053 \times 10^6 \text{ m/s}$$

and $z_e = \dfrac{\omega}{q v_{Fe}}$

$v_{Fi} = v_{Fe} \dfrac{m_e}{m_i}$ and $z_i = z_e \dfrac{m_i}{m_e}$

$(z_i \gg z_e)$

PLASMONS

We want $\varepsilon_c(\vec{q},\omega) = 0$; i.e.

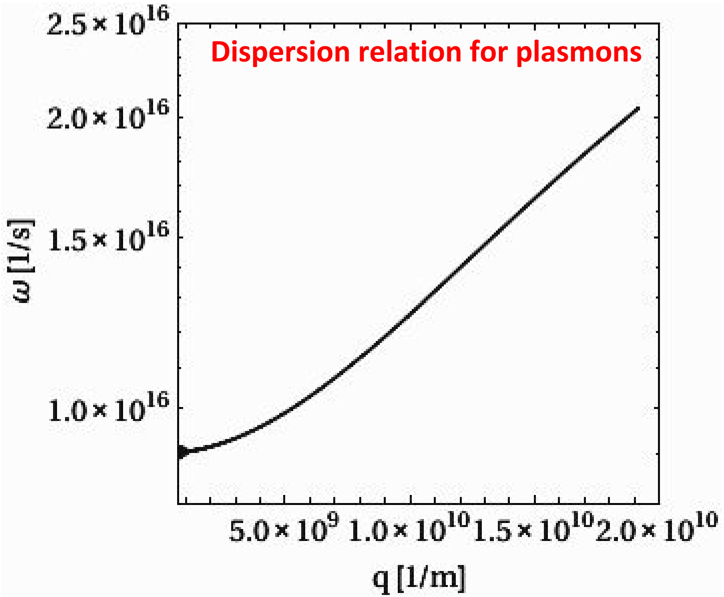
$$-q^2 = k_i^2\left[2 - z_i \ln\frac{z_i+1}{z_i-1}\right] + k_e^2\left[2 - z_e \ln\frac{z_e+1}{z_e-1}\right]$$

(1) Pick z_e (real and >1); $z_i = z_e \frac{m_i}{m_e}$

(2) Calculate $-q^2$

(3) $\omega = z_e\, q\, v_{Fe}$

(4) Result is (q, ω) pair.



Orders of magnitude

$q \sim 10^{10}\ m^{-1} \approx \dfrac{1}{\text{atomic radius}}$

$\hbar\omega \sim \hbar \times 10^{16} s \approx 6.6\ eV$

These quasiparticles are quantized plasma waves, called plasmons.

PHONONS

There is another solution of $\varepsilon_c(\vec{q},\omega) = 0$ with $z_e \ll 1$, but $z_i = \frac{m_i}{m_0} z_e \gg 1$.

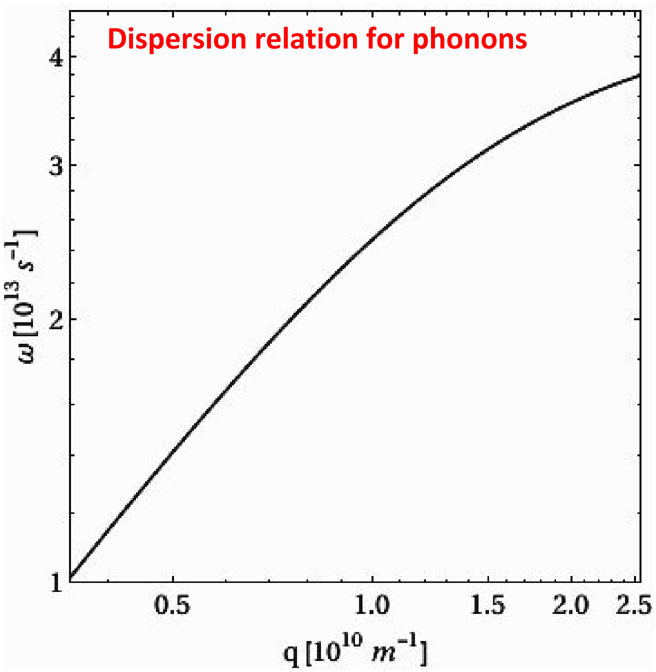
$$-q^2 = K_i^2\left[2 - z_i \ln\frac{z_i+1}{z_i-1}\right] + K_e^2 \underbrace{\left[2 - z_e \ln\frac{z_e+1}{z_e-1}\right]}_{\approx\, 2 - z_e\, i\pi}$$

Remember, $z_i = \dfrac{\omega_1 + i\omega_2}{q\, v_{Fi}} = z_{i1} + i z_{i2}$

So unlike the plasmons, the phonons are damped waves. To get the frequency ω_1 (approximately)

$$-q^2 = K_i^2\left[2 - z_{1i} \ln\frac{z_{1i}+1}{z_{1i}-1}\right] + 2K_e^2$$

(the damping is small)



For long wavelengths ($q \to 0$),

$\omega = c_{sound}\, q$;

according to Harris,

$c_{sound} = v_{Fe}/\sqrt{3} \times (m_e/m_i)^{1/2}$

Homework Problems due Wednesday March 2

Problem 28.
Use the dielectric function that was derived in the lecture of Feb. 22 to calculate the electrostatic potential $\Phi(\mathbf{x})$ about a stationary charge Q immersed in the plasma.
Hints: the potential would be Q/r if there were no plasma; for a stationary charge the frequency ω is 0.

Problem 29.
Calculate the speed of sound in metallic sodium, based on the theory of phonons described in the lecture of Feb. 22. Compare the result to the measured value.