Forget about particles.

What are the fields?

What equations govern the fields?

★ We always start with a classical field theory.

★ The field equations come from Lagrangian dynamics.

Today's example: The Lagrangian for the Schroedinger equation.

Review of Lagrangian dynamics

For a single coordinate q(t) :

Lagrangian L = L (q, dq/dt) ;

and Action A = $\int_{t1}^{t2}$ L(q , dq/dt) dt .

The equation of motion for q(t) comes from the requirement that δA = 0 (with endpoints fixed); i.e., the action is an extremum. For a variation δq(t)

$$\delta A = 0 = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot q}\,\frac{d}{dt}(\delta q) \right] dt$$

integrate by parts; $\delta q = 0$ at t_1 and t_2

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) \right] \delta q \, dt$$

must be zero for any variation $\delta q(t)$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0$$

(Lagrange's equation)

Canonical momentum and the Hamiltonian

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$H = p\dot{q} - L$$

which must be rewritten in terms of p and q

<u>Example.</u> A particle in a potential...

$$q(t) = \vec{x}(t)$$

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x})$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = \frac{d}{dt}(m\dot{\vec{x}}) + \frac{\partial V}{\partial \vec{x}}$$

$$\ddot{\vec{x}} = \vec{F}/m \quad \text{where } \vec{F} = -\partial V/\partial \vec{x}$$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}}$$

$$H = \vec{p}\cdot\dot{\vec{x}} - L = \frac{p^2}{m} - \frac{p^2}{2m} + V(\vec{x})$$

$$H = \frac{p^2}{2m} + V(\vec{x})$$

Canonical Quantization (Dirac)

Rules to convert classical dynamics to a quantum theory:

★ q and p become operators; they operate on the Hilbert space of physical states.

★ $[\, q , p \,] = i \hbar$

★ H is the generator of translation in time.

Theorem.
H is the generator of translation in time for the quantum theory.
Suppose $L = \frac{1}{2} M (dq/dt)^2 - V(q)$.

$$H = \frac{p^2}{2m} + V(q)$$

- $$\frac{i}{\hbar}[H,q] = \frac{i}{\hbar}\frac{1}{2m}[p^2, q] = \frac{i}{2m\hbar}\{ p[p,q] + [p,q]p \} = \frac{i}{2m\hbar}(-i\hbar p)2 = p/m$$ = dq/dt
- $$\frac{i}{\hbar}[H,p] = \frac{i}{\hbar}[V(q), p] = \frac{i}{\hbar}\, i\hbar \frac{\partial V}{\partial q} = -\frac{\partial V}{\partial q}$$ = dp/dt

Q.E.D.

So far, we have considered only one degree of freedom. Now consider a system with many degrees of freedom; $\{ q_i : i = 1\ 2\ 3 \ldots D \}$

For many degrees of freedom…

$$q(t) \longrightarrow Q(t) \equiv \{ q_i(t)\ ;\ i = 1\ 2\ 3 \ldots i \ldots N \}$$

- $L = L(Q, dQ/dt)$ ⇒ Lagrange's equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$ for $i = 1\ 2\ 3 \ldots N$

- Canonical momentum… $\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$
- and Hamiltonian… $\boxed{H = \sum_{i=1}^{N} p_i \dot{q}_i - L}$

(which *must be* re-expressed in terms of $p_1 \ldots p_N$ and $q_1 \ldots q_N$.)

Classical field theory

(suppress spin for now)

We replaced

$q(t) \rightarrow \{ q_i(t) \;;\; i \in Z \}$; *discrete*

Now replace

$q(t) \rightarrow \{ \psi(\mathbf{x},t) \;;\; \mathbf{x} \in R^3 \}$; *continuum*

◈

$L = L(\psi(\mathbf{x},t), \partial\psi(\mathbf{x},t)/\partial t)$

$L = \int \mathfrak{L}(\psi(\mathbf{x},t), \nabla\psi(\mathbf{x},t), \partial\psi(\mathbf{x},t)/\partial t)\, d^3x$

Lagrange's equation ---

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\psi}(x)}\right) - \frac{\delta L}{\delta \psi(\vec{x})} = 0$$

functional derivatives;

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)}\right) - \frac{\partial \mathcal{L}}{\partial \psi(\vec{x})} = 0$$

partial derivatives;

this is the *"classical field theory."*

Mathematically it's an example continuum dynamics.

THE LAGRANGIAN FOR SCHROEDINGER WAVE MECHANICS

$$A = \int_{t_1}^{t_2} dt \int d^3x \left\{ \frac{-i\hbar}{2}\left(\frac{\partial \psi^*}{\partial t}\psi - \psi^* \frac{\partial \psi}{\partial t}\right) - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - V\psi^*\psi \right\}$$

Note: A is real.

But $\psi(x)$ is complex.

We could write $\psi = R + iI$

and treat R and I as independent

"generalized coordinates."

$\Rightarrow$ δR and δI variations. w/ $\delta A = 0$.

Easier, and equivalent, treat ψ and ψ^*

as independent "generalized coordinates"

$\Rightarrow$ $\delta\psi$ and $\delta\psi^*$ variations w/ $\delta A = 0$.

Lagrange's Equations

$$A = \int_{t_1}^{t_2} dt \int d^3x \left\{ -\frac{i\hbar}{2}\left(\frac{\partial \psi^*}{\partial t}\psi - \psi^* \frac{\partial \psi}{\partial t} \right) - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - V\psi^*\psi \right\}$$

Vary A w.r.t. ψ^* keeping ψ fixed

$$\delta A = \int_{t_1}^{t_2} dt \int d^3x \left\{ -\frac{i\hbar}{2}\left[\frac{\partial}{\partial t}(\delta\psi^*)\psi - \delta\psi^* \frac{\partial \psi}{\partial t} \right] - \frac{\hbar^2}{2m} \nabla\delta\psi^* \cdot \nabla\psi - V\,\delta\psi^*\,\psi \right\}$$

Integrate by parts; the surface terms are 0.
(b/c $\delta\psi^* = 0$ at t_1 and t_2; and $\psi^* = 0$ at ∞.)

$$\delta A = \int_{t_1}^{t_2} dt \int d^3x \left\{ +\frac{i\hbar}{2}\,\delta\psi^*\,\dot{\psi} + \frac{i\hbar}{2}\,\delta\psi^*\,\dot{\psi} + \frac{\hbar^2}{2m}\,\delta\psi^*(\nabla^2\psi) - V\,\delta\psi^*\,\psi \right\}$$

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

The other Lagrange equation, from $\delta\psi(x)$ keeping ψ^* fixed, is

$$-i\hbar\frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*$$

Thus, the classical field equation is the Schroedinger equation.

Canonical momenta

$$\pi(x) = \frac{\delta L}{\delta \dot{\psi}(x)} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)} = \frac{i\hbar}{2}\psi^*(x)$$

$$\pi^*(x) = \frac{\delta L}{\delta \dot{\psi}^*(x)} = -\frac{i\hbar}{2}\psi(x)$$

The Hamiltonian

$$H = \int d^3x \left(\pi\,\dot{\psi} + \pi^*\,\dot{\psi}^* \right) - L$$

$$= \int d^3x \left\{ \frac{i\hbar}{2}\psi^*\dot{\psi} - \frac{i\hbar}{2}\psi\dot{\psi}^* + \frac{i\hbar}{2}\dot{\psi}^*\psi - \frac{i\hbar}{2}\psi^*\dot{\psi} + \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi \right\}$$

$$= \int d^3x \left\{ \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi \right\}$$

Quantization

So far, this is the classical field theory. Now…

Dirac's canonical commutation relation $[\, q \,, p \,] = i\hbar$ is valid for Hermitian operators q and p. We need to change that (because ψ is complex) to

$$[\psi(x), \pi(x')] = \frac{i\hbar}{2}\,\delta^3(x-x') \quad \text{where } \pi = \frac{i\hbar}{2}\psi^\dagger$$
$$[\psi^\dagger(x), \pi^\dagger(x')] = \frac{i\hbar}{2}\,\delta^3(x-x')$$

Therefore

$$[\psi(x), \psi^\dagger(x')] = \delta^3(x-x')$$
$$[\psi(x), \psi(x')] = 0$$

Or, replace these by *anticommutators* for fermions.

Summary

$$[\, \psi(\mathbf{x}) \,, \psi(\mathbf{x}') \,] = 0$$
$$[\, \psi(\mathbf{x}) \,, \psi^\dagger(\mathbf{x}') \,] = \delta^3(\mathbf{x}-\mathbf{x}') \quad ;$$

or, use *anticommutators* for fermions;

$$\boxed{H = \int d^3x \left\{ \frac{\hbar^2}{2m} \nabla\psi^\dagger \cdot \nabla\psi + V\psi^\dagger\psi \right\}}$$

This is precisely the NRQFT that we have been using, but with a 1-body potential $V(\mathbf{x})$ and without a 2-body potential $V_2\,(\mathbf{x,y})$.

Exercise: Figure out the Lagrangian that would include a 2-body potential. Hint: The Lagrangian must include a term quartic in the field.

Exercise: Verify that H is the generator of translation in time, in the quantum theory.

Homework Problems due Wednesday March 2

Problem 30. Equal time commutation relations. We have, in the Schroedinger picture,

$$[\, \psi(\mathbf{x}) \, , \, \psi^{\dagger}(\mathbf{x}') \,] = \delta^3(\mathbf{x}-\mathbf{x}') \quad ,$$

etc.

(a) Show that in the Heisenberg picture, this commutation relation holds at all equal times.

(b) What is the commutation relation for different times?

Problem 31.

(a) Do problem 2.1 in Mandl and Shaw.

(b) Do problem 2.2 in Mandl and Shaw.

(c) Do problem 2.3 in Mandl and Shaw.