The classical field theory describes electromagnetic waves with $\omega = ck$.

The quantum field theory describes photons. (Chapter 1)

We can derive the theory from a Lagrangian, and then quantize it. But there are some subtleties, due to gauge invariance! (Chapter 5)

Electromagnetism isn’t very interesting without sources, i.e., charges.

We’ll add the electron field in PHY 955. That’s Quantum ElectroDynamics. (Ch. 7)
Now another example: (SECTION 2.2 - 2.3)

A REAL SCALAR FIELD $\varphi = \varphi(x,t)$

This example is relativistically covariant.

\[ L = \frac{1}{2} \left( \frac{\partial \varphi}{\partial t} \right)^2 - \frac{c^2}{2} (\nabla \varphi)^2 - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \varphi^2 \]

Derive the field equation from Hamilton's principle,

\[ A = \int \left\{ \frac{1}{2} \dot{\varphi}^2 - \frac{c^2}{2} (\nabla \varphi)^2 - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \varphi^2 \right\} dx dt \]

\[ \delta A = \int \left\{ \dot{\varphi} (\delta \varphi) - c^2 \nabla \varphi \cdot \nabla (\delta \varphi) - \left( \frac{mc^2}{\hbar} \right)^2 \delta \varphi \right\} dx dt \]

\[ = \int \delta \varphi \left\{ -\dot{\varphi} + c^2 \nabla^2 \varphi - \left( \frac{mc^2}{\hbar} \right)^2 \varphi \right\} dx dt \]

\[ = 0 \text{ for any variation } \delta \varphi. \text{ Therefore} \]

\[ \ddot{\varphi} - c^2 \nabla^2 \varphi + \left( \frac{mc^2}{\hbar} \right)^2 \varphi = 0 \]

the Klein-Gordon equation

Recall the example of the Schrödinger equation

Classical field theory: $\psi(x,t)$ is a complex function.

\[ A = \int_t^{t_2} dt \int d^3x \left\{ -\frac{i\hbar}{2} \left( \frac{\partial \psi}{\partial t} - \psi^* \frac{\partial \psi}{\partial x} \right) - \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi - \nabla^2 \psi \right\} \]

Quantum: $\psi(x,t)$ is a non-hermitian operator.

\[
\begin{align*}
\left[ \psi(x), \psi^+(x') \right] &= \delta^3(x-x') \\
\left[ \psi(x), \psi(x') \right] &= 0
\end{align*}
\]
We can solve the Klein-Gordon equation, in plane waves,

\[ \phi(x,t) = C e^{i(k \cdot x - \omega t)} \]

where

\[ -\omega^2 + c^2 k^2 + \left(\frac{mc^2}{\hbar}\right)^2 = 0 \]

\[ \omega = \pm \sqrt{c^2 k^2 + \frac{m^2 c^4}{\hbar^2}} \]

I.e.,

\[ \frac{\hbar}{c} \omega = \pm \sqrt{c^2 \hbar^2 k^2 + m^2 c^4} \]

Note that this is the energy (\(\hbar \omega\)) and momentum (\(\hbar \mathbf{k}\)) relation of special relativity.

(What are the negative energy solutions?)

The general solution (Hermitian) is

\[ \phi(x,t) = \sum \frac{N}{k} \left\{ e^{i(k \cdot x - \omega t)} a^+_k + e^{-i(k \cdot x - \omega t)} a_k \right\} \]

Quantization

We can anticipate

\[ [ a_k, a_{k'}^\dagger ] = \delta_k(\mathbf{k}, \mathbf{k}') \]

\[ [ a_k, a_{k'} ] = 0 \]

Derive this from Dirac’s canonical quantization. Recall,

\[ [ q, p ] = i \hbar \quad \text{where} \quad p = \partial L/\partial q' \]

(What are the negative energy solutions?)
The Hamiltonian

\[ H = \pi \dot{\phi} - L \]

rewritten in terms of \( \phi, \pi \)

\[ H = \int \pi(x) \dot{\phi}(x) \, d^3x - L \]

rewritten in terms of \( \phi(x), \pi(x) \)

Homework problem.

(A) Write \( H \) in terms of \( \pi(x) \) and \( \phi(x) \).

(B) Write \( H \) in terms of \( a_k \) and \( a_k^\dagger \).

Homework problem.

Determine the Green's function for the free scalar field; \( <0 | T \phi(x) \phi(y) | 0 > \).

\[ \text{Infinite volume limit: } \frac{1}{\mu} = \frac{1}{3 \pi^2} \text{Vol} \]

\[ N^2 \approx 2.5, N = \sqrt{\frac{\hbar}{2 \omega m}} \]
Next: A real scalar field $\phi$ with a source $\rho$.

The particular solution for a static source

Consider $\rho = \rho(x)$, independent of $t$.

The field equation is a linear inhomogeneous equation; so $\phi(x,t) = \phi_{\text{particular}}(x,t) + \phi_{\text{homogeneous}}(x,t)$.

The particular solution comes from the source; e.g., it could be a mean field produced by a static source; or, waves radiated by a time dependent source.

The homogeneous solution consists of harmonic waves.

The particular solution for a static source

Consider $\rho = \rho(x)$, independent of $t$.

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 + \rho \phi$$

Field equation $\mathcal{L} \phi = 0$;

$$\Rightarrow \quad \phi - \nabla^2 \phi + m^2 \phi - \rho = 0$$

We need the Green's function of $\nabla^2 + m^2$; i.e.,

$$(-\nabla^2 + m^2) G(x-y) = \delta^3(x-y)$$

Then

$$\phi_0(x) = \int G(x-y) \rho(y) \, dy$$
The Green’s function of \(-\nabla^2 + m^2\)

\[
(-\nabla^2 + m^2) G(\mathbf{\xi}) = \delta^3(\mathbf{\xi})
\]

\[
G(\mathbf{\xi}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{\xi}}}{k^2 + m^2}
\]

\[
= \frac{1}{2\pi^2} \frac{1}{2\pi} \int_0^\infty \frac{k^2 dk}{k^2 + m^2} \int^\infty_{-1} d\cos \theta \ e^{ik_3 \cos \theta}
\]

\[
= \frac{1}{4\pi^2} \frac{2\pi}{i\xi} \frac{1}{2i\xi} \sum \frac{kdk}{(k-im)(k+im)} \ e^{ik_3}
\]

\[
\xi > 0 \text{ so close the contour above}
\]

\[
= \frac{1}{4\pi^2} \frac{2\pi}{i\xi} \frac{1}{2i\xi} e^{-m\xi}
\]

\[
= \frac{e^{-m|\xi|}}{4\pi|\xi|} \quad (\text{W/ } \xi = 1 \text{ and } c=1)
\]

Example

Suppose \(\rho(x) = \rho_o \theta(a-r)\).

\[
\phi_0(x) = \int \frac{e^{m|x-y|}}{4\pi|x-y|} \theta(a-y) \, dy
\]

Limiting cases —

- Large \(r\)
  \[
  \phi_0(r) \sim \frac{J_0}{4\pi} \frac{e^{-mr}}{r} \cdot \frac{y}{8\pi a^3}
  \]

- Small \(r\)
  \[
  \phi_0(r) \sim \frac{J_0}{4\pi} \int \frac{e^{-my}}{y} \theta(a-y) \, dy
  \]
  \[
  = \frac{J_0}{m^2} \left\{ 1 - (1+ma)e^{-ma} \right\} \quad \text{Still } \lambda = 1 \text{ and } c=1.
  \]
The interaction Lagrangian density
\[ L_{\text{interaction}} = g \, \Psi_{\alpha \rho}^\dagger \, \Psi_{\alpha \rho} \, \phi \]

- This \( L_{\text{int}} \) acts as a source for \( \phi \), with
  \[ \rho (x,t) = g \, \Psi_{\alpha \rho}^\dagger \, \Psi_{\alpha \rho} \cdot \]

- It also acts as a potential for \( \Psi \):
  \[ V_{\text{int}} (x,t) = - g \, \phi (x,t) \cdot \]

∴ The field equations;
  i.e., Lagrange’s equations,

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + (-g \phi) \psi = i \hbar \frac{\partial \psi}{\partial t} \]

\[ \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = g \, \psi^\dagger \psi \]
Calculate the potential energy for a nucleon (N) attracted to a heavy isotope (Z,A)

First step -- calculate the mean field created by the nucleons in the heavy isotope.

\[-\nabla^2 \phi + m^2 \phi = \langle \phi \psi \psi \rangle\]

\[\langle \psi \psi \rangle_{U-238} = \text{density of nucleons} = \sum_{n=1}^{238} |\psi_n(x)|^2\]

\[\simeq \frac{A}{\frac{1}{A^3} \chi_{R_y}^2} \Theta(R-y) \text{ where } R = \gamma A^\frac{1}{3}\]

\[\phi(x) = \int G(x-y) n(y) d^3 y\]

Second step -- calculate the potential energy for the presence of the extra nucleon.

\[V(x) = -g \phi = -g \int G(x-y) n(y) d^3 y\]

\[V(x) = \frac{-3}{4\pi \gamma_r^2 A} \int \frac{e^{-\gamma_r |x-y|}}{4\pi |x-y|^2} \Theta(\gamma_r A^{\frac{1}{3}} - |y|) d^3 y\]

\[h = 1 \text{ and } c = 1\]

Rewrite this for numerical calculation...
Yukawa’s theory of the nucleon-nucleon force (1935)

1. Nucleons interact through a scalar field $\phi$ with mass $m$.

2. The range of the force is

$$\text{range} = \frac{\hbar}{mc} = 1 \text{ to } 2 \text{ fm}$$

$$\therefore \quad mc^2 = \frac{\hbar c}{\text{range}} = 100 \text{ to } 200 \text{ MeV}$$

Of course Yukawa did not know about pions, which were discovered in 1947.

- mass ($\pi^\pm$) = 139.6 MeV/c$^2$
- mass ($\pi^0$) = 135.0 MeV/c$^2$

The Lagrangian density for the theory is

$$\mathcal{L} = \mathcal{L}_{\text{nucleon}} + \mathcal{L}_{\text{meson}} + \mathcal{L}_{\text{interaction}}$$

Nucleon field = $\psi_{\alpha \rho}(x)$

$\alpha =$ spin index and $\rho =$ isospin index

$$\psi_{\alpha \rho}(x) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{k,s,t} e^{i(kx - \omega t)} \eta_{\alpha} \eta_{\rho} a_{kst}$$

Meson field = $\phi(x)$ with isospin 0 to follow Yukawa

Lagrange’s equations including the interaction, $\mathcal{L}_{\text{interaction}} = g \, \psi^{\dagger}_{\alpha \rho} \psi_{\alpha \rho} \phi$:

- $-\frac{\hbar^2}{2m} \nabla^2 \psi + (-g\phi) \psi = i\hbar \frac{\partial \psi}{\partial t}$

- $\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi + m^2 \phi = g \psi^{+} \psi$
Numerical calculations

\[ r_0 = 1.25 \text{ fm} \quad R = r_0 A^{1/3} \]
\[ mc^2 = 140 \text{ MeV} \quad \text{pion mass} \]
\[ A = 238 \quad \text{uranium} \]
\[ g = 15 \quad \text{strong interaction} \]

The potential energy for the extra nucleon is

\[ V(r) = -g^2 \phi_0(r). \]
Homework due Wednesday, March 2

Problem 32.
For the free real scalar field,
(A ) Write $H$ in terms of $\pi(x)$ and $\varphi(x)$.
(B ) Write $H$ in terms of $a_k$ and $a_k^\dagger$.

Problem 33.
(A ) Mandl and Shaw problem 3.3.
(B ) Mandl and Shaw problem 3.4.