Phy 955 Homework Set 1 (Mar. 25)

Problem 1 The antiparticle spinors are

(A) $v_1(\vec{p}) = N\begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E+m \end{pmatrix}$, $v_2(\vec{p}) = N\begin{pmatrix} p_z \\ p_x+ip_y \\ E+m \\ 0 \end{pmatrix}$

where $N = \dfrac{1}{\sqrt{2m(E+m)}}$ and $E = \sqrt{\vec{p}^2+m^2}$.

(B) Spin sum : $\Lambda^-(\vec{p}) = -\sum_{s=1}^{2} v_s(\vec{p})\,\bar{v}_s(\vec{p}) = \dfrac{-(\not{p}-m)}{2m}$

Problem 2 (A) Mandl & Shaw problem 4.3

Let $s^\mu(x) = -e\,\bar{\psi}\gamma^\mu\psi(x)$ = charge current density (operator)

Now calculate

$[\,s^\mu(x), s^\nu(y)\,]$. Use identities :
$$\begin{cases} [A,BC] = [A,B]C + B[A,C] \\ [A,BC] = \{A,B\}C - B\{A,C\} \\ \text{etc.} \end{cases}$$

Result

$$[s^\mu(x), s^\nu(y)] = e^2\,\bar{\psi}(x)\gamma^\mu\, iS(x-y)\,\gamma^\nu\psi(y) - e^2\,\bar{\psi}(y)\gamma^\nu\, iS(y-x)\,\gamma^\mu\psi(x)$$

where $iS(x-y) = \{\psi(x), \bar{\psi}(y)\} = (i\gamma\cdot\partial + m)\,\Delta(x-y)$

Now recall Eq. (3.49) ; $\Delta(x-y) = 0$ for $(x-y)^2 < 0$.

Thus $[s^\mu(x), s^\nu(y)] = 0$ for $(x-y)^2 < 0$.

Problem 2 (B) Mandl & Shaw problem 4.4

Consider $\phi(x) = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega\Omega}} \left[a_{\vec{k}} e^{-ik\cdot x} + a^{+}_{\vec{k}} e^{ik\cdot x} \right]$

with anticommutators ; $\{a_{\vec{k}}, a^{+}_{\vec{k}'}\} = \delta_{\vec{k}\vec{k}'}$ and $\{a_{\vec{k}}, a_{\vec{k}'}\} = 0$.

Now calculate ① $[\phi(x), \phi(y)]$ and ② $\{\phi(x), \phi(y)\}$ for $(x-y)^2 < 0$.

② $\{\phi(x), \phi(y)\} = \sum_{\vec{k}} \frac{1}{2\omega\Omega} \left[e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)} \right]$

$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \boxed{\cos} \left[k_\mu (x-y)^\mu \right]$, which is not 0 for $(x-y)^2 < 0$.

① $[\phi(x), \phi(y)]$ is obviously not 0 for $(x-y)^2 < 0$, because it includes terms like

$$\sum_{\vec{k}} \frac{1}{\sqrt{2\omega\Omega}} \sum_{\vec{k}'} \frac{1}{\sqrt{2\omega'\Omega}} \left[a_{\vec{k}}, a_{\vec{k}'} \right] e^{-ik\cdot x} e^{-ik'\cdot y} ;$$

Note $[a_{\vec{k}}, a_{\vec{k}'}] = a_{\vec{k}} a_{\vec{k}'} - a_{\vec{k}'} a_{\vec{k}} = 2 a_{\vec{k}} a_{\vec{k}'}$

which annihilates 2 electrons ; and there is no other term in $[\phi(x), \phi(y)]$ to cancel this.

Comment: $\int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \sin\left[k_\mu (x-y)^\mu \right] = 0$ if $(x-y)^2 < 0$.

proof: if $(x-y)^2 < 0$ consider $x^0 = y^0$; $\frac{1}{\omega} \sin(-\vec{k} \cdot (\vec{x} - \vec{y}))$ is an odd function of $\vec{k}$.

Problem 3 Mandl & Shaw problem 4.5

- Chiral phase transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x)$$

$$\psi^\dagger(x) \to \psi'^\dagger(x) = \psi^\dagger(x)\, e^{-i\alpha\gamma_5}$$

where $\gamma_5 = \frac{i}{4}\epsilon_{\lambda\mu\nu\rho}\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho = \gamma^5$

In the Dirac representation of the gamma matrices,

$$\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \gamma^k = \begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix} \text{ and } \gamma_5 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Note that $\{\gamma^\mu, \gamma_5\} = 0$ and $\gamma_5^2 = 1$.

- The Lagrangian density $\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial - m)\psi$ ($\hbar = 1$ and $c = 1$) is invariant if $m = 0$.

Proof $\mathcal{L}' = \bar{\psi}'(i\gamma\cdot\partial - m)\psi' = \underbrace{\psi'^\dagger}_{\psi^\dagger e^{-i\alpha\gamma_5}}\gamma^0(i\gamma\cdot\partial - m)\underbrace{\psi'}_{e^{i\alpha\gamma_5}\psi}$

$$= \bar{\psi}\,\gamma^0 e^{-i\alpha\gamma_5}\gamma^0(i\gamma\cdot\partial - m)\, e^{i\alpha\gamma_5}\psi$$

$e^{-i\alpha\gamma_5}\gamma^0 = \gamma^0 e^{+i\alpha\gamma_5}$ because $\gamma_5\gamma^0 = -\gamma^0\gamma_5$

$e^{i\alpha\gamma_5}\gamma^\mu = \gamma^\mu e^{-i\alpha\gamma_5}$ because $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$

$$\mathcal{L}' = \bar{\psi}\gamma^0\gamma^0(i\gamma\cdot\partial - m\, e^{2i\alpha\gamma_5})\psi$$

$$= \bar{\psi}(i\gamma\cdot\partial - m)\psi + m\bar{\psi}\psi(1 - e^{2i\alpha\gamma_5})$$

$\mathcal{L}' = \mathcal{L}$ if $m = 0$

- The corresponding conserved current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi \quad \text{(see page 32)}$$

where $\psi' = e^{i\alpha\gamma_5}\psi = (1 + i\alpha\gamma_5)\psi = \psi + \delta\psi$ (infinitesimal α)

$$\Rightarrow j^\mu = \bar{\psi} i\gamma^\mu i\alpha\gamma_5 \psi = -\alpha \, \bar{\psi}\gamma^\mu\gamma_5\psi$$

Thus The conserved current is the <u>axial</u> <u>vector</u> <u>current</u>

$j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi$; it is conserved if $m=0$.

- Now define $\frac{1}{2}(1 \pm \gamma_5)\psi = \begin{cases} \psi_R \\ \psi_L \end{cases}$

The field equation for ψ_L is

$$\begin{aligned} i\gamma\cdot\partial\psi_L &= \frac{i}{2}(\gamma\cdot\partial\psi - \gamma^\mu\gamma_5\partial_\mu\psi) \\ &= \frac{i}{2}(\gamma\cdot\partial\psi + \gamma_5\gamma\cdot\partial\psi) = \frac{1}{2}(1+\gamma_5)(i\gamma\cdot\partial\psi) \\ &= \frac{1}{2}(1+\gamma_5)m\psi = m\psi_R . \end{aligned}$$

If $m=0$ then $i\gamma\cdot\partial\psi_L$ so ψ_L and ψ_R are decoupled.

- $\mathcal{L} = \bar{\psi}_L i\gamma\cdot\partial\psi_L$ describes $m=0$ particles, with helicity $= -1$, because $\Pi^\pm(\vec{p}) = \frac{1}{2}(1 \pm \frac{\sigma\cdot\vec{p}}{p}) = \frac{1}{2}(1 \pm \gamma_5)$ (equations A.37 and A.43) for $m=0$ particles.