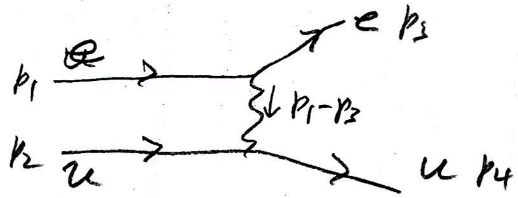
Homework Assignment #4

Problem 13.



The matrix element is just like $e\mu$ scattering but with charge $+\frac{2}{3}e$ for the u quark

$$\mathcal{M} = \frac{-2}{3}e^2 \; \bar{u}_3 \gamma^\mu u_1 \; \bar{u}_4 \gamma_\mu u_2 \; / \; (p_1 - p_3)^2$$

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{4}{9}\frac{e^4}{t^2}\frac{1}{(2m)^4} \mathrm{Tr}\, \gamma^\mu (\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m) \qquad m = \text{electron mass}$$

$$\mathrm{Tr}\, \gamma_\mu(\not{p}_2 + M)\gamma_\nu(\not{p}_4 + M) \qquad M = \text{u quark mass}$$

$$= \frac{1}{9}\frac{e^4}{t^2}\frac{1}{(2m)^4}\left(4 p_1^\mu p_3^\nu - 4 g^{\mu\nu} p_1\cdot p_3 + 4 p_3^\mu p_1^\nu\right) \qquad \text{neglecting masses}$$

$$\left(4 p_{2\mu} p_{4\nu} - 4 g_{\mu\nu} p_2\cdot p_4 + 4 p_{4\mu} p_{2\nu}\right)$$

$$= \frac{16}{9}\frac{e^4}{t^2}\frac{1}{(2m)^4}\Big[2\, p_1\cdot p_2\; p_3\cdot p_4 + 2 p_1\cdot p_4\; p_2\cdot p_3 - p_1\cdot p_3\; 2 p_2\cdot p_4$$

$$- p_2\cdot p_4\; 2 p_1\cdot p_3 + 4\; p_1\cdot p_3\; p_2\cdot p_4\Big]$$

Neglecting masses, $s = (p_1 + p_2)^2 = 2 p_1\cdot p_2 = (p_3 + p_4)^2 = 2 p_3\cdot p_4$

$$t = (p_1 - p_3)^2 = -2 p_1\cdot p_3 = (p_4 - p_2)^2 = -2 p_4\cdot p_2$$

$$u = (p_1 - p_4)^2 = -2 p_1\cdot p_4 = (p_3 - p_2)^2 = -2 p_3\cdot p_2$$

$$\overline{|\mathcal{M}|^2} = \frac{16}{9}\frac{e^4}{t^2}\frac{1}{(2m)^4}\left[2(s/2)^2 + 2(u/2)^2\right]$$

$$\left(\frac{d\sigma}{d\Omega}\right)_c = \frac{1}{64\pi^2 s}(2m)^4\, \overline{|\mathcal{M}|^2} = \frac{e^4}{8\cdot 9\pi^2}\; \frac{s^2 + u^2}{s t^2}$$

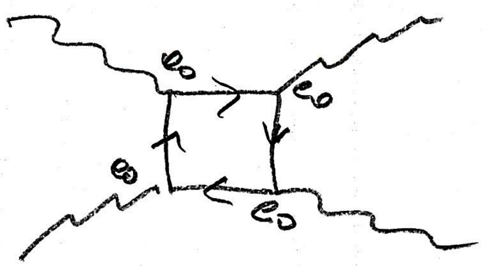
4 points

2 points

$$\text{P.D.G.} = \frac{2}{9}\frac{\alpha^2}{s}\left(\frac{s^2+u^2}{t^2}\right) \quad \text{because } e^2 = 4\pi\alpha$$

6 total

Problem 14



$\mathcal{M} \propto e_0^4$ so $d\sigma \propto \alpha^4 \sim 10^{-8}$

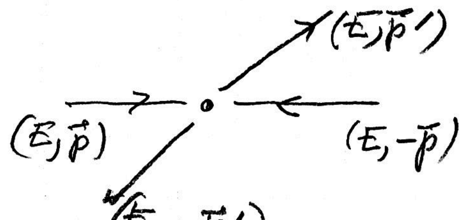
The cross section is too small to measure.

2 points

Problem 15

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{16E^4}\frac{p'}{E}\left(E^2 + M^2 + p'^2\cos^2\theta\right)$$

(lepton pair production in e^+e^- collision, neglecting the electron mass)



where $E = \sqrt{\vec{p}^{\,2} + m_e^2} = \sqrt{\vec{p}\,'^2 + M^2}$

$p' = \sqrt{E^2 - M^2}$

center of mass energy $= 2E$

$$\sigma_{tot} = \frac{\pi\alpha^2}{4E^4}\left(\frac{p'}{E}\right)\left(E^2 + M^2 + \frac{1}{3}p'^2\right)$$

$$\frac{dP}{d\Omega} = \frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{1}{4\pi}\,\frac{E^2 + M^2 + p'^2\cos^2\theta}{E^2 + M^2 + p'^2/3}$$ with $p' = \sqrt{E^2 - M^2}$

Consider $E = 110$ MeV, 300 MeV, 1 GeV ; $M = 106$ MeV

3 points

Problem 15

In[25]:=
```
M = 106.0
δP[θ_] := (Ee^2 + M^2 + (Ee^2 - M^2) * Cos[θ]^2) /
      (Ee^2 + M^2 + (Ee^2 - M^2) / 3) / (4 * Pi)
Plot[
  {δP[θ] /. {Ee → 110},
    δP[θ] /. {Ee → 300},
    δP[θ] /. {Ee → 1000}}, {θ, 0, Pi},
  PlotRange → {{0, Pi}, {0, 0.2}},
  Frame → True, FrameLabel → {"q", "dP/dΩ"}]
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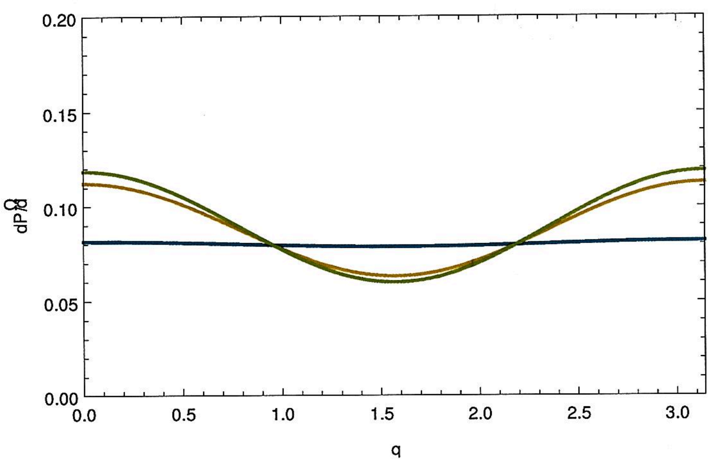
Out[25]= 106.

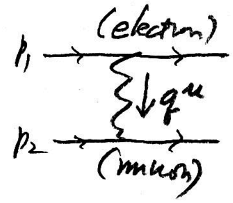
Out[27]=



Problem 16

(4 points total for (A))

(A) Mandl & Shaw problem 8.2. Consider $e\mu$ scattering; neglect the electron mass

Matrix element: $\mathcal{M} =$ 

$= e^2 \bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu u_2 / t$

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{e^4}{t^2}\frac{1}{(2m)^4}\left(4p_1^\mu p_3^\nu - 4g^{\mu\nu} p_1\cdot p_3 + 4p_3^\mu p_1^\nu\right)$$
$$\left(4p_{2\mu}p_{4\nu} - 4g_{\mu\nu}\, p_2\cdot p_4 + 4p_{4\mu}p_{2\nu} + 4M^2 g_{\mu\nu}\right)$$

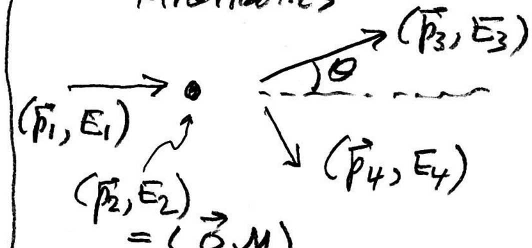
$$= \frac{1}{4}\frac{e^4}{t^2}\frac{32}{(2m)^4}\left(p_1\cdot p_2\; p_3\cdot p_4 + p_1\cdot p_4\; p_3\cdot p_2 - M^2 p_1\cdot p_3\right)$$

$$s = (p_1+p_2)^2 = M^2 + 2p_1\cdot p_2 = (p_3+p_4)^2 = M^2 + 2p_3\cdot p_4$$
$$t = (p_1-p_3)^2 = -2p_1\cdot p_3 = (p_4-p_2)^2 = 2M^2 - 2p_2\cdot p_4$$
$$u = (p_1-p_4)^2 = M^2 - 2p_1\cdot p_4 = (p_3-p_2)^2 = M^2 - 2p_3\cdot p_2$$

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}\frac{1}{(2m)^4}\left[(s-M^2)^2 + (M^2-u)^2 + 2M^2 t\right]$$

Lab frame of reference ($\vec{p}_2 = 0$)

Kinematics



$$q^2 = t = -2p_1\cdot p_3$$
$$= -2p_1 p_3(1-\cos\theta)$$
$$= -4E_1E_3 \sin^2\theta/2$$

$$\frac{d\sigma}{d\Omega_3} = \frac{(2m)^4\overline{|M|^2}}{64\pi^2 v\, E_1E_2E_3E_4}\delta(E_3+E_4-E_1-E_2)\, p_3^2\, dp_3$$

$$= \frac{(2m)^4\overline{|M|^2}\, p_3^2}{64\pi^2 p_1 M p_3 E_4}\underbrace{\delta(p_3+E_4-p_1-M)\,dp_3}_{\frac{1}{\left|\frac{\partial}{\partial p_3}(p_3+E_4)\right|}}$$

$$= \frac{(2m)^4\overline{|M|^2}}{64\pi^2}\;\frac{p_3}{p_1 M(E_4 + p_3 - p_1\cos\theta)}$$

$$E_4 = \sqrt{M^2 + (\vec{p}_1-\vec{p}_3)^2}$$
$$\frac{\partial E_4}{\partial p_3} = \frac{1}{E_4}(p_3 - p_1\cos\theta)$$

So far: $\overline{|\mathcal{M}|^2}(2m)^4 = \frac{2e^4}{t^2}\left[(s-M^2)^2+(M^2-u)^2+2M^2t\right]$

and $\frac{d\sigma}{d\Omega_3} = \frac{(2m)^4\overline{|\mathcal{M}|^2}}{64\pi^2}\;\frac{p_3}{p_1 M(M+2p_1\sin^2\theta/2)}$

Now express everything in lab frame variables

$$s = M^2 + 2p_1\cdot p_2 = M^2+2ME_1$$

$$t = q^2 = 2M^2-2p_2\cdot p_4 = 2M^2-2ME_4 = -2p_1\cdot p_3$$
$$= -2p_1p_3(1-\cos\theta)$$
$$= -4E_1E_3\sin^2\theta/2$$

$$u = M^2-2p_3\cdot p_2 = M^2-2ME_3$$

$$\overline{|\mathcal{M}|^2}(2m)^4 = \frac{2e^4}{16E_1^2E_3^2\sin^4\frac{\theta}{2}}\left[4M^2E_1^2+4M^2E_3^2+2M^2(2M^2-2ME_4)\right]$$

Recall the Mott cross section (with $Z=1$) $\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2(1-\sin^2\theta/2)}{4E_1^2\sin^4\theta/2}$

$$\overline{|\mathcal{M}|^2}(2m)^4 = \frac{2}{16E_3^2}\;\frac{4(4\pi)^2}{\cos^2\theta/2}\left(\frac{d\sigma}{d\Omega}\right)_{Mott}4M^2(E_1^2+E_3^2+M^2-ME_4)$$

Thus

$$\left(\frac{d\sigma}{d\Omega_3}\right)_L = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\frac{1}{64\pi^2}\;\frac{32\pi^2M^2}{E_3^2\cos^2\theta/2}\;\frac{(E_1^2+E_3^2+M^2-ME_4)\,p_3}{p_1M(M+2p_1\sin^2\theta/2)}$$

Further simplifications $\left(\frac{d\sigma}{d\Omega_3}\right)_L = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\frac{M(E_1^2+E_3^2+M^2-ME_4)}{2p_3\cos^2\frac{\theta}{2}\,p_1(M+2p_1\sin^2\theta/2)}$

$$E_1^2+E_3^2+M^2-ME_4 = E_1^2+E_3^2+M(M-E_4) = E_1^2+E_3^2+M(p_3-p_1)$$
$$= p_1^2-Mp_1+p_3^2+Mp_3$$
$$= p_1^2+p_3^2-2p_1p_3\sin^2\theta/2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_L = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\frac{M(p_1^2+p_3^2-2p_1p_3\sin^2\theta/2)}{2\cos^2\theta/2\;p_1p_3(M+2p_1\sin^2\frac{\theta}{2})}$$

where $Mp_1 = (M+2p_1\sin^2\frac{\theta}{2})\,p_3$

Energy Conservation

$$p_1+M = p_3+\sqrt{M^2+(\vec{p}_1-\vec{p}_3)^2}$$
$$\therefore M^2+p_1^2+p_3^2-2p_1p_3\cos\theta = p_1^2+2Mp_1+M^2+p_3^2-2Mp_3-2p_1p_3$$
$$-2p_1p_3\cos\theta = 2M(p_1-p_3)-2p_1p_3$$
$$M(p_1-p_3) = 2p_1p_3\sin^2\theta/2$$

It still needs a bit more work.

① Note that $p_3 = \frac{M p_1}{M + 2p_1 \sin^2\theta/2} = p_1 \left[1 + \frac{2E_1}{M} \sin^2\theta/2 \right]^{-1}$

So

$$\left(\frac{d\sigma}{d\Omega}\right)_L = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left[1 + \frac{2E_1}{M} \sin^2\frac{\theta}{2}\right]^{-1} \frac{p_1}{p_3} \frac{M(p_1^2 + p_3^2 - 2p_1 p_3 \sin^2\theta/2)}{2\cos^2\frac{\theta}{2}\, p_1 p_3 (p_3 - p_1 \cos\theta)}$$

call this Σ and simplify.

② We are supposed to show that

$$\Sigma = 1 - \frac{q^2}{2M^2} \tan^2\theta/2 \quad \text{where} \quad q^2 = -4E_1 E_3 \sin^2\frac{\theta}{2}$$

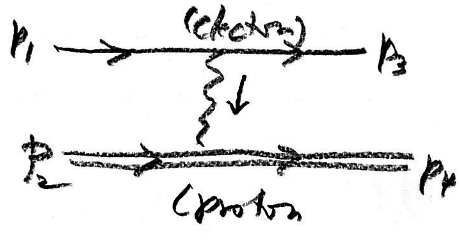
Since we already know the answers we can just use Mathematica to verify that the two expressions are equal. Note: $M(p_1 - p_3) = 2p_1 p_3 \sin^2\theta/2$ and $q^2 = -4p_1 p_3 \sin^2\theta/2$

Part B: ← Total points = 4

Problem 16 part (B)

(Mandl & Shaw #8.3)

The matrix element for e-proton scattering



(similar to $e\mu$ scattering)

$f_1 = F_1(q^2)$

$f_2 = \frac{K}{2M} F_2(q^2)$

$$\mathcal{M} = \frac{e^2}{t} \bar{u}_3 \gamma^\mu u_1 \, \bar{u}_4 \left(\gamma_\mu f_1 + i\sigma_{\mu\alpha} q^\alpha f_2 \right) u_2$$

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \frac{e^4}{t^2} \operatorname{Tr} \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_3 \bar{u}_3$$

$$\operatorname{Tr} \Gamma_\mu u_2 \bar{u}_2 \Gamma_\nu u_4 \bar{u}_4$$

$$= \frac{1}{4} \frac{e^4}{t^2} E^{\mu\nu} P_{\mu\nu}$$

$\Gamma_\mu = \gamma_\mu f_1 + i\sigma_{\mu\alpha} q^\alpha f_2$

$\sigma_{\mu\alpha} = \frac{i}{2}[\gamma_\mu, \gamma_\alpha]$

The electron tensor is ($m_e = 0$)

$$E^{\mu\nu} = \frac{1}{(2m)^2} \operatorname{Tr} \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3 = \frac{1}{(2m)^2} \left\{ 4 p_1 p_3 - 4 g\, p_1 \cdot p_3 + 4 p_3 p_1 \right\}^{\mu\nu}$$

Note that $q_\mu E^{\mu\nu} = 0$ and $q_\nu E^{\mu\nu} = 0$.

In the calculation of the proton tensor we can drop any term $\propto q_\mu$ or q_ν.

$$(2M)^2 P_{\mu\nu} = f_1^2 \operatorname{Tr} \gamma_\mu (\not{p}_2 + M) \gamma_\nu (\not{p}_4 + M)$$ ← exactly the same as $e\mu$ scattering $\times f_1^2$

$$+ f_2^2 \left(\tfrac{i}{2}\right)^2 \operatorname{Tr} [\gamma_\mu, \not{q}] (\not{p}_2 + M) [\gamma_\nu, \not{q}] (\not{p}_4 + M)$$

$+ f_1 f_2$ terms

The contribution to $d\sigma/d\Omega$ from the terms $\propto f_1^2$ is exactly the same as in problem 8.2 (for $e\mu$ scattering)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{proton}} = f_1^2\left(\frac{d\sigma}{d\Omega}\right)_{\text{muon}} + f_2^2\left(\frac{d\sigma}{d\Omega}\right)_{22} + f_1 f_2\left(\frac{d\sigma}{d\Omega}\right)_{12}$$

For the contribution $\propto f_2^2$ we have

$$|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{t^2}\left(\frac{-f_2^2}{4}\right)E^{\mu\nu}\,\text{Tr}\,[\gamma_\mu, \not{q}](\not{p}_2+M)[\gamma_\nu,\not{q}](\not{p}_4+M)$$

Note $$[\gamma_\mu, \not{q}] = \{\gamma_\mu, \not{q}\} - 2\not{q}\gamma_\mu = 2q_\mu - 2\not{q}\gamma_\mu$$

$$\text{(Tr)} = 4\,\text{Tr}\,(q_\mu - \not{q}\gamma_\mu)(\not{p}_2+M)(q_\nu - \not{q}\gamma_\nu)(\not{p}_4+M)$$

but $q_\mu E^{\mu\nu}=0$ and $q_\nu E^{\mu\nu}=0$

$$= 4\,\text{Tr}\,\not{q}\gamma_\mu(\not{p}_2+M)\not{q}\gamma_\nu(\not{p}_4+M)$$

$$= 4\,\text{Tr}\,\not{q}\gamma_\mu\not{p}_2\not{q}\gamma_\nu\not{p}_4 + 4M^2\,\text{Tr}\,\not{q}\gamma_\mu\not{q}\gamma_\nu$$

(trace $\gamma^{\text{odd}} = 0$)

$$= 4\{0 - q\cdot p_2\,\text{Tr}\,\gamma_\mu\not{q}\gamma_\nu\not{p}_4 + q^2\,\text{Tr}\,\gamma_\mu\not{p}_2\gamma_\nu\not{p}_4 - 0 + q\cdot p_4\,\text{Tr}\,\gamma_\mu\not{p}_2\not{q}\gamma_\nu\} + 4M^2\{0 - 4q^2 4g_{\mu\nu}^{+0}\}$$

(dropping terms $\propto e_\mu$ or e_ν)

$$= -4q\cdot p_2(-4g_{\mu\nu}\,q\cdot p_4) + 4q^2(4p_{2\mu}p_{4\nu} - 4g_{\mu\nu}p_2\cdot p_4 + 4p_{2\nu}p_{4\mu}) + 4q\cdot p_4(+4g_{\mu\nu}\,q\cdot p_2) - 64M^2q^2g_{\mu\nu}$$

$L_{\mu\nu}$ same as for a lepton

$$\overline{|\mathcal{M}|^2}_{(22)} = \frac{e^4}{4t^2}\left(\frac{-f_2^2}{4}\right)\Big\{ E^{\mu\nu} g_{\mu\nu}\left[32\, q\cdot p_2\, q\cdot p_4 - 64 M^2 q^2\right] + 4q^2 E^{\mu\nu} L_{\mu\nu} \Big\}$$

$$= \frac{e^4}{4t^2}(-f_2^2 q^2) E^{\mu\nu} L_{\mu\nu}$$

← Same as for a lepton; $f_2 = \frac{K}{2M} F_2$

$$+ \frac{e^4}{4t^2}\left(\frac{-f_2^2}{4}\right)(8\, p_1\cdot p_3 - 16\, p_1\cdot p_3)(32\, q\cdot p_2\, q\cdot p_4 - 64 M^2 q^2)$$

$= -4q^2$

$(t = (p_1 - p_3)^2 = -2 p_1\cdot p_3 = q^2)$

$$= \frac{e^4}{4t^2}(-f_2^2 q^2) E^{\mu\nu} L_{\mu\nu} + \frac{e^4}{4t^2}(f_2^2 q^2)(32\, q\cdot p_2\, q\cdot p_4 - 64 M^2 q^2)$$

$$q\cdot p_2 = (p_1 - p_3)\cdot p_2 = (p_4 - p_2)\cdot p_2 = p_2\cdot p_4 - M^2$$

$(t = (p_1 - p_3)^2 = (p_4 - p_2)^2 = 2M^2 - 2 p_2\cdot p_4)$

$$= (M^2 - t/2) - M^2 = -t/2$$

$$q\cdot p_4 = (p_4 - p_2)\cdot p_4 = M^2 - p_2\cdot p_4 = +t/2$$

$$\overline{|\mathcal{M}|^2}_{(22)} = \frac{e^4}{4t^2}(-f_2^2 q^2) E^{\mu\nu} L_{\mu\nu} + \frac{e^4}{4t^2}(f_2^2 q^2)(-8t^2 - 64 M^2 q^2)$$

Just the same as for a lepton × $(-f_2^2 q^2)$; also proportional to $(-f_2^2 q^2)$

$$\therefore f_2^2 \left(\frac{d\sigma}{d\Omega}\right)_{22} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left(1 + \frac{2E}{M}\sin^2\frac{\theta}{2}\right)^{-1}\left[1 - \frac{q^2}{2M^2}\tan^2\theta\right]\left(\frac{-K^2}{2M^2}F_2^2 q^2\right)$$

$$+ \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left(1 + \frac{2E}{M}\sin^2\theta/2\right)^{-1}(+8q^4 + 64 M^2 q^2)\left(\frac{-K^2}{2M^2}F_2^2 q^2\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{proton} = F_1^2\left(\frac{d\sigma}{d\Omega}\right)_{muon} + \left(\frac{-K^2}{4M^2}F_2^2 q^2\right)\left(\frac{d\sigma}{d\Omega}\right)_{muon} + \text{another term} \propto F_2^2 q^2$$

$$+ \text{a term} \propto f_1 f_2$$