Mandl and Shaw, Chapter 4
Notations, conventions and units

Section 1.2: RATIONALIZED gaussian electromagnetic units
Section 2.1: Relativity notations
Section 6.1: Natural units

$$x^{\mu} = (x^0, \mathrm{x})\ ;\ x^0 = c\,t$$

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$$

$$x_{\mu} = g_{\mu\nu}\, x^{\nu}\ ; \qquad (x_0, x_i) = (x^0, -x^i)$$

$$\mathrm{p.x} = g_{\mu\nu}\, p^{\mu} x^{\nu} = p^0 x^0 - \mathrm{p.x}$$

$$p^2 = \mathrm{p}\cdot\mathrm{p} = E^2 - |\,p\,|^2 = m^2$$

$$\partial_{\mu} = \partial / \partial x^{\mu} = (\partial / \partial x^0,\ \nabla)$$

$$\hbar = 1 \text{ and } c = 1$$

$$\alpha = e^2 / (4\pi\hbar c) = 1/137.037$$

The Dirac Equation
(Chapter 4; Appendix A)

/1/ Recall the Schroedinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

The plane wave solutions

$$\Psi(\boldsymbol{x},t) = C\, e^{i(\mathbf{p}.\boldsymbol{x} - E\,t)} \qquad (\hbar = 1)$$

$$H\,\Psi = E\,\Psi \ \Rightarrow\ E = p^2/2m \qquad \text{(nonrelativisic)}$$

$$\mathbf{P} = -i\,\nabla, \qquad \text{so} \quad \mathbf{P}\,\Psi = \mathbf{p}\,\Psi(\boldsymbol{x},t).$$

The plane wave is an eigenstate of momentum.

/2/ The Dirac equation

We want an equation that is (i) linear in time, (ii) with plane wave solutions, (iii) such that $E = \sqrt{p^2+m^2}$

$$\Psi(x,t) \propto e^{i(\mathbf{p}.\mathbf{x} - Et)}$$

$$i\,\partial\Psi/\partial t = H\,\Psi$$

Should we try

$$H\,\Psi = \sqrt{p^2+m^2}\;\Psi$$

i.e., $\quad H = \sqrt{P^2+m^2} \quad ?$

But that is a nonlocal operator.

To be consistent with relativity, *t* and *(x, y, z)* should be treated similarly;
because the Lorentz transformations mix *t* and *(x, y, z)*. So let's try

$$i\,\partial\Psi/\partial t = (\boldsymbol{\alpha}.\mathbf{P} + \beta m)\,\Psi$$

...with $\quad (\boldsymbol{\alpha}.\mathbf{P} + \beta m)^2 = P^2 + m^2$

The quantities β and $(\alpha_x, \alpha_y, \alpha_z)$ will be ***matrices***.

Now try

$$\Psi(\boldsymbol{x},t) \propto e^{i(\mathbf{p}.\mathbf{x} - Et)}\,u$$

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\, u = E\, u$$

$$(\alpha\cdot\vec{p} + \beta m)^2\, u = E^2 u$$

$$\{ \alpha_i \alpha_j p^i p^j + \beta^2 m^2 + m p^i(\alpha^i\beta + \beta\alpha^i)\}\, u = (\vec{p}^{\,2} + m^2)\, u$$

So we must have

$$\tfrac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i) = \delta_{ij}$$

$$\beta\alpha^i + \alpha^i\beta = 0$$

$$\beta^2 = 1$$

β and (α_x , α_y , α_z)

Since they don't commute, they must be matrices.

Four - vector notations (Appendix A)

Define $\gamma^0 = \beta$;

also, $(\gamma^1, \gamma^2, \gamma^3) = (\beta\alpha_x, \beta\alpha_y, \beta\alpha_z)$

UPPER AND LOWER INDICES:

$\{x^0, x^1, x^2, x^3\} = \{ct, x, y, z\}$ $(c = 1)$

$\{x_0, x_1, x_2, x_3\} = \{ct, -x, -y, -z\}$

$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\} = \beta\{1, \alpha_x, \alpha_y, \alpha_z\}$

$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} = \beta\{1, -\alpha_x, -\alpha_y, -\alpha_z\}$

$\gamma\,.\,A = \gamma^\mu A_\mu = \gamma_\mu A^\mu = \gamma^0 A^0 - \gamma^i A^i$

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\nabla\psi + \beta m\psi$$

$$i\gamma^0\frac{\partial\psi}{\partial x^0} = -i\vec{\gamma}\cdot\nabla\psi + m\psi$$

$$i\left(\gamma^0\frac{\partial}{\partial x^0} + \vec{\gamma}\cdot\nabla\right)\psi - m\psi = 0$$

$$i\,(\gamma^0\ \partial/\partial x^0 + \gamma^i\,\partial/\partial x^i\,)\,\psi\ -\mathrm{m}\ \psi\ = 0$$

That is the Dirac equation.

Various notations may be used

$$\mathrm{i}\ \gamma^\mu\,\frac{\partial\psi}{\partial x^\mu} - \mathrm{m}\ \psi\ = 0$$

$$\mathrm{i}\ \gamma\,.\,\partial\ \psi - \mathrm{m}\ \psi\ = 0$$

$$\mathrm{i}\ \not\partial\ \psi - \mathrm{m}\ \psi = 0 \qquad \text{defines } \not Q = \gamma^\mu\,Q_\mu$$

/3/ The gamma matrices

What are the gamma matrices?

They are not unique.

The gamma matrices are 4 X 4 matrices, defined by certain anticommutation relations:

$$\{\,\alpha^i,\,\alpha^j\,\} = 2\ \delta_{\mathrm{ij}}$$

$$\{\,\gamma^i\ ,\ \gamma^j\,\} = \{\,\beta\alpha^i\ ,\ \beta\alpha^j\,\} = -\ 2\ \delta_{\mathrm{ij}}$$

$$\{\,\gamma^i\ ,\ \gamma^0\,\} = \{\,\beta\alpha^i\ ,\ \beta\,\} = 0$$

$$\{\,\gamma^0\ ,\ \gamma^0\,\} = 2\,(\gamma^0)^2 = 2$$

Thus, the defining equation is

$$\{\,\gamma^\mu\,,\,\gamma^\nu\,\} = 2\ g^{\mu\nu} \qquad\qquad (\star)$$

⊟ *The standard representation* ("Dirac rep.") *for the gamma matrices is*

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and σ_i = ith Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(we never raise the index on the Pauli matrices!)

Exercise. Verify (★).

⊟ *The Majorana representation*

$$\gamma_M^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_M^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} \quad \text{etc}$$

(A.79)

which is sometimes convenient.

(Peskin and Schroeder use yet a different representation.)

Theorem. If $\{ \gamma^\mu , \gamma^\nu \} = 2\, g^{\mu\nu}$, and U is a unitary matrix ($U^\dagger U = 1$), then $\{ \gamma'^\mu , \gamma'^\nu \} = 2\, g^{\mu\nu}$ where $\gamma'^\mu = U \gamma^\mu U^\dagger$.

Proof.

$$\{\gamma'^\mu, \gamma'^\nu\} = U\gamma^\mu U^\dagger U \gamma^\nu U^\dagger + U\gamma^\nu U^\dagger U \gamma^\mu U^\dagger$$
$$= U \{\gamma^\mu, \gamma^\nu\} U^\dagger$$
$$= 2 g^{\mu\nu}\, U U^\dagger = 2 g^{\mu\nu}$$

Exercise. Find U such that

$$\gamma_M{}^\mu = U \gamma^\mu U^\dagger .$$

For most calculations, we don't need to use any specific representation of the gamma matrices. Instead we can use some identities that are true for all representations.

$$\{ \gamma^\mu , \gamma^\nu \} = 2\, g^{\mu\nu} \qquad (\bigstar)$$

/4/ Examples of gamma matrix identities

- Trace ($\gamma^\mu \gamma^\nu$)

Lemma. Trace(BA) = Trace(AB).

Proof.
Trace(BA) = $B_{rs} A_{sr} = A_{sr} B_{rs}$ = Trace(AB).
Even if A and B do not commute,
i.e., BA $\neq$ AB, always Tr(BA) = Tr(AB).

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu = \tfrac{1}{2} \mathrm{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$
$$= \tfrac{1}{2} \mathrm{Tr}\ 2 g^{\mu\nu} \cdot 1 = 4 g^{\mu\nu}$$

- Trace ($\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$)

$$\mathrm{Tr}\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$
$$= \mathrm{Tr} (\{ \gamma^\mu ; \gamma^\nu \} \gamma^\rho \gamma^\sigma) - \mathrm{Tr}\ \gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma$$
$$= 2 g^{\mu\nu}\ 4 g^{\rho\sigma} - \mathrm{Tr} (\gamma^\nu \{ \gamma^\mu ; \gamma^\rho \} \gamma^\sigma) + \mathrm{Tr} (\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma)$$
$$= 8 g^{\mu\nu} g^{\rho\sigma} - 8\, g^{\mu\rho} g^{\nu\sigma} + \underbrace{\mathrm{Tr} (\gamma^\nu \gamma^\rho \{ \gamma^\mu ; \gamma^\sigma \})}_{8 g^{\mu\sigma} g^{\nu\rho}} - \underbrace{\mathrm{Tr}\ \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu}_{\text{is equal to } \mathrm{Tr}\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma}$$

$$\mathrm{Tr}\ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4 g^{\mu\nu} g^{\rho\sigma} - 4 g^{\mu\rho} g^{\nu\sigma} + 4 g^{\mu\sigma} g^{\nu\rho}$$

- $\gamma^\mu \gamma^\rho \gamma_\mu$

$$\gamma^\mu \gamma^\rho \gamma_\mu = \{\gamma^\mu, \gamma^\rho\}\gamma_\mu - \gamma^\rho \gamma^\mu \gamma_\mu$$
$$= 2g^{\mu\rho}\gamma_\mu - \gamma^\rho \cdot 4$$
$$= -2\gamma^\rho$$

Etc.

We'll use many such identities.
See the Appendix ; A.2 and A.3.

/5/ The Dirac spinors

- Plane wave solutions of the Dirac equation

$$(i\not{\partial} - m)\psi = 0$$

$$\psi_\alpha(x) = \underbrace{e^{i(\vec{p}\cdot\vec{x} - Et)}}_{e^{-ip\cdot x}} u_\alpha(\vec{p}, \lambda)$$

where $p\cdot x = p^0 x^0 - \vec{p}\cdot\vec{x} = p_\mu x^\mu$

$$[i\gamma^\mu(-ip_\mu) - m]\,u = 0$$

$$(\not{p} - m)\,u = 0$$

u_r is an eigenvector of $\cancel{p}$

Now, consider $(\not{p}-m)(\not{p}+m)$

$$= \not{p}\not{p} - m^2 = \gamma^\mu\gamma^\nu p_\mu p_\nu - m^2$$

$$= p^2 - m^2$$

$$= 0$$

Therefore, $u(\vec{p},\lambda)$ can be any of the 4 columns of $\not{p}+m$.

- In the standard (Dirac) representation:

$$\not{p} + m = \gamma^0 E - \gamma^i p^i + m$$

$$= \begin{bmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{bmatrix}$$ in 2×2 block form

The first two colums are

$$\begin{pmatrix} E+m & 0 \\ 0 & E+m \\ p^3 & p^1 - ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix}$$

Thus the particle spinors are

$$u(\vec{p},1) = N\begin{pmatrix} E+m \\ 0 \\ p^3 \\ p^1+ip^2 \end{pmatrix} \text{ and } u(\vec{p},2) = N\begin{pmatrix} 0 \\ E+m \\ p^1-ip^2 \\ -p^3 \end{pmatrix}$$

The other 2 solutions are antiparticle spinors.

Normalization choice

This can be done in different ways.
We'll follow Mandl and Shaw ; eq. (A.27) ;

$$u_{\mathbf{r}}^{\dagger}(\mathbf{p})\, u_{\mathbf{s}}(\mathbf{p}) = \frac{E}{m}\,\delta_{rs} \qquad w/\ \ r \text{ and } s \in \{1,2\}$$

$$v_{\mathbf{r}}^{\dagger}(\mathbf{p})\, v_{\mathbf{s}}(\mathbf{p}) = \frac{E}{m}\,\delta_{rs} \qquad w/\ \ r \text{ and } s \in \{1,2\}$$

Also, define $\bar{u} = u^{\dagger}\gamma^0$ and $\bar{v} = v^{\dagger}\gamma^0$;

Then

$$\bar{u}_{\mathbf{r}}(\mathbf{p})\, u_{\mathbf{s}}(\mathbf{p}) = \delta_{rs} \qquad \text{and} \qquad \bar{v}_{\mathbf{r}}(\mathbf{p})\, v_{\mathbf{s}}(\mathbf{p}) = -\,\delta_{rs}$$

$\Rightarrow$ Completeness relation

$$\sum_{r} \left[\, u_r \bar{u}_r - v_r \bar{v}_r \,\right] = \mathbf{1}_{\mathbf{4x4}}$$

Check:

$$u_1 = N\begin{pmatrix} E+m \\ 0 \\ p_z \\ p_x+ip_y \end{pmatrix} \quad \& \quad u_1^{\dagger} = N\left(E+m \quad 0 \quad p_z \quad p_x-ip_y\right)$$

$$u_1^{\dagger} u_1 = N^2\left[\,(E+m)^2 + p_z^2 + p_x^2+p_y^2\,\right]$$
$$= N^2\left[\,E^2+2Em + m^2 + \vec{p}^{\,2}\,\right]$$
$$= N^2\ 2E(E+m)$$
$$= \frac{E}{m} \quad \Rightarrow \quad N = \frac{1}{\sqrt{2m(E+m)}}$$

Now

$$\bar{u}_1 u_1 = u_1^{\dagger}\gamma^0 u_1 = u_1^{\dagger}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_1$$
$$= N^2\left[\,(E+m)^2 - p_z^2 - p_x^2 - p_y^2\,\right]$$
$$= N^2\left[\,E^2+2Em+m^2-\vec{p}^{\,2}\,\right]$$
$$= N^2\ 2m(E+m) = 1 \qquad \checkmark$$

Polarization sums. Mandl&Shaw call these "energy projection operators" ; Section A.5 ;

$\Lambda^{\pm}(p) = (\pm\not{p}+m)/2m$ *[note: these are 4X4 matrices]*

$$\Lambda^{+}(p) = \sum_{r} u_r(p)\,\bar{u}_r(p) \quad \text{and} \quad \Lambda^{-}(p) = -\sum_{r} v_r(p)\,\bar{v}_r(p)$$

Homework Problems due Friday, March 25

Problem 1.

A. Determine the Dirac spinors $v_1(p)$ and $v_2(p)$ for antiparticles.

B. Determine the polarization sum $\Lambda^{-}(p)$ for antiparticles.