Quantization of the Dirac Field (Section 4.3)

First quantization, for the Dirac equation,

$$(i\gamma \cdot \partial - m)\, \psi = 0 .$$

The plane wave solutions are

$$e^{-ip.x}\, u_s(p) \qquad s \in \{1,2\}$$

$$\boldsymbol{p.x} = p^0x^0 - \mathbf{p.x}$$

$$p^0 = \sqrt{\mathbf{p}^2+m^2} \equiv E_p$$

(positive energy solutions)

$$e^{+ip.x}\, v_s(p) \qquad \text{(negative energy solutions)}$$

Spinor definitions :

$$(\gamma \cdot p - m)\, u_s(p) = 0 \qquad s \in \{1,2\}$$

$$(\gamma \cdot p + m)\, v_s(p) = 0$$

$\bar{u}\, u = 1$ and $\bar{v}\, v = -1$ where $\bar{u} = u^{\dagger} \gamma^0$.

Now, we can expand $\psi(x)$ in plane waves,

$$\psi(x) = \sum_{\vec{p}\, s} \left(\frac{m}{\Omega E_p} \right)^{1/2} \left\{ c_s(\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x} + d_s^{+}(\vec{p})\, v_s(\vec{p})\, e^{ip\cdot x} \right\}$$

and

$$\bar{\psi}(x) = \sum_{\vec{p}\, s} \left(\frac{m}{\Omega E_p} \right)^{1/2} \left\{ c_s^{+}(\vec{p})\, \bar{u}_s(\vec{p})\, e^{ip\cdot x} + d_s(\vec{p})\, \bar{v}_s(\vec{p})\, e^{-ip\cdot x} \right\}$$

The coefficients $c_s(\mathbf{p})$ and $d_s(\mathbf{p})$ will become annihilation operators.

① *Second quantization -- creation and annihilation operators*

(familiar from PHY 855)

$$\{ c_r(\vec{p}), c_s^\dagger(\vec{p}') \} = \delta_{rs}\, \delta_{\vec{p},\vec{p}'}$$

$$\{ d_r(\vec{p}), d_s^\dagger(\vec{p}') \} = \delta_{rs}\, \delta_{\vec{p},\vec{p}'}$$

all other anti-commutators vanish.

E.g., $\{c_r, c_s\} = 0$; $\{c_r, d_s^\dagger\} = 0$, etc.

② *The equal time anticommutation relations (E.T.aC.R.)*

$\{\psi_\alpha(x), \psi_\beta(y)\}$ with $t_x = t_y$ $(x^0 = y^0)$

$\propto \{ c \text{ and } d^\dagger, c \text{ and } d^\dagger \}$

$= 0$

$\{\psi_\alpha(x), \psi_\beta^\dagger(y)\}$ with $x^0 = y^0$

$$= \sum_{\vec{p}s} \sum_{\vec{p}'s'} \frac{m}{\Omega\sqrt{EE'}} \{ c_s(p) u_{s\alpha}(p) e^{-ip\cdot x} + d_s^\dagger(p) v_{s\alpha}(p) e^{ip\cdot x},$$
$$c_{s'}^\dagger(p') \bar{u}_{s'\beta'}(p') e^{ip'\cdot y} + d_{s'}(p') \bar{v}_{s'\beta'}(p') e^{-ip'\cdot y} \} \gamma^0_{\beta'\beta}$$

$\Rightarrow$ $\{c_s(p), c_{s'}^\dagger(p')\} = \delta_{ss'}\,\delta(\vec{p},\vec{p}')$

$\{d_s^\dagger(p), d_{s'}(p')\} = \delta_{ss'}\,\delta(\vec{p},\vec{p}')$

$E = E'$ so $e^{-ip\cdot x} e^{ip'\cdot y} = e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$

$e^{ip\cdot x} e^{-ip'\cdot y} = e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$

$$= \sum_{\vec{p}s} \frac{m}{\Omega E} \{ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} u_{s\alpha}(p) \bar{u}_{s\beta'}(p) + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} v_{s\alpha}(p) \bar{v}_{s\beta'}(p) \} \gamma^0_{\beta'\beta}$$

$$= \sum_{p} \frac{m}{\Omega E} \{ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \Lambda^+_{\alpha\beta'}(p) + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} (-)\Lambda^-_{\alpha\beta'}(p) \} \gamma^0_{\beta'\beta}$$

$$\Lambda^+(p) = \frac{\not{p}+m}{2m} = \frac{\gamma^0 E_p - \vec{\gamma}\cdot\vec{p} + m}{2m}$$

$$\Lambda^-(p) = \frac{-\not{p}+m}{2m} = \frac{-\gamma^0 E_p + \vec{\gamma}\cdot\vec{p} + m}{2m}$$

In the second line, let $\vec{p} \to -\vec{p}$ (because we sum over $\vec{p}$)

$$= \sum_{p} \frac{m}{\Omega E} \{ e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \} \left[\frac{1}{2m}\right] [E_p - \vec{\gamma}\cdot\vec{p}\gamma^0 + m\gamma^0 + E_p \mp \gamma\cdot\vec{p}\gamma^0 - m\gamma^0]_{\alpha\beta}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} 1_{\alpha\beta} = \delta^3(\vec{x}-\vec{y})\, \delta_{\alpha\beta} \quad (*)$$

③ *Canonical quantization*

/3a/ We start with a Lagrangian ...

$$L = \int \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi \; d^3x$$

Let's check the field equations:

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \dot{f}}\right) - \frac{\delta L}{\delta f} = 0$$

Variation of $\bar{\psi}$: $\quad 0 - (i\gamma^\mu\partial_\mu - m)\,\psi = 0$

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

Variation of ψ:

$$\frac{\partial}{\partial t}\left(\bar{\psi}\, i\gamma^0\right) - \left(-i\nabla\bar{\psi}\cdot\vec{\gamma} - m\bar{\psi}\right) = 0$$

$$i\frac{\partial}{\partial x^\mu}\left(\bar{\psi}\gamma^\mu\right) + m\bar{\psi} = 0$$

↑ equiv. to adjoint of Dirac eq.

/3b/ …from which we can derive the commutation relation;
Dirac method of quantization;

$$L = \int \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \, d^3x$$

The canonical momentum π conjugate to ψ is

$$\pi = \frac{\delta L}{\delta \dot{\psi}} = \bar{\psi} i\gamma^0 = i\psi^\dagger$$

So the Dirac quantization rule is

$$\{ \psi(\vec{x}t), \pi(\vec{y}t) \} = i\delta^3(\vec{x}-\vec{y})$$

or $\{ \psi(\vec{x},t), \psi^\dagger(\vec{y}t) \} = \delta^3(\vec{x}-\vec{y})$

which agrees with (*)

④ *The Feynman propagator* = $S_F(x\!-\!y)$

(Section 4.4)

Recall from PHY 855 — we want the propagator for the *time-ordered product of fields*. (*Do you remember why?*)

⊗ This is the definition of $S_F(\boldsymbol{x}-\boldsymbol{y})$:

$$i\, S_F(\boldsymbol{x}-\boldsymbol{y}) = \langle 0|\ T\, \psi(\boldsymbol{x})\, \overline{\psi}(\boldsymbol{y})\ |0\rangle$$

$$= \begin{cases} \langle 0|\ \psi(\boldsymbol{x})\, \overline{\psi}(\boldsymbol{y})\ |0\rangle & \text{if } x^0 > y^0 \\ -\langle 0|\ \overline{\psi}(\boldsymbol{y})\, \psi(\boldsymbol{x})\ |0\rangle & \text{if } x^0 < y^0 \end{cases}$$

⊗ This is the formula for $S_F(\boldsymbol{x}-\boldsymbol{y})$, as a Fourier integral:

$$S_F(\boldsymbol{x}-\boldsymbol{y}) = \int \frac{d^4p}{(2\pi)^4}\, \frac{\gamma.p + m}{p^2 - m^2 + i\varepsilon}\, e^{-i\,\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})}$$

$(\ \varepsilon \rightarrow 0+\ )$

$S_F(\boldsymbol{x}-\boldsymbol{y})$ is "*the Green's function with Feynman boundary conditions*". Recall $\Delta_F(\boldsymbol{x}-\boldsymbol{y})$.

⑤ *Derivation of the Fourier integral*

First, calculate $\langle 0|\, \psi(x)\, \overline{\psi}(y)\, |0\rangle$, using the creation and annihilation operators

$$= \sum_{\vec{p}s} \sum_{\vec{p}'s'} \left(\frac{m}{\Omega E}\right)^{1/2} \left(\frac{m}{\Omega E'}\right)^{1/2}$$

$$\langle 0|\left[\, c_s(\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x} + d_s^\dagger(\vec{p})\, v_s(\vec{p})\, e^{ip\cdot x} \right]$$

$$\left[\, c_{s'}^\dagger(\vec{p}')\, \bar{u}_{s'}(\vec{p}')\, e^{ip'\cdot y} + d_{s'}(\vec{p}')\, \bar{v}_{s'}(\vec{p}')\, e^{-ip'\cdot y} \right]|0\rangle$$

4 terms ; 3 are 0 ; $\langle 0|\, c_s(\vec{p})\, c_{s'}^\dagger(\vec{p}')\,|0\rangle = \delta_{ss'}\, \delta_{\vec{p}\vec{p}'}$

$$= \sum_{\vec{p}s} \frac{m}{\Omega E}\, u_s(p)\, \bar{u}_s(p)\, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, \frac{m}{E}\, \Lambda^+(p)\, e^{-ip\cdot(x-y)} \quad ; \quad \Lambda^+(p) = \frac{\not{p}+m}{2m}$$

$$= (i\not{\partial} + m) \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2E}\, e^{-ip\cdot(x-y)}$$

$$= (i\not{\partial} + m)\ i\Delta^+(x-y) \qquad [\text{see Eq. } 3.39]$$

Also, similarly,

$$-\langle 0|\, \overline{\psi}(y)\, \psi(x)\, |0\rangle = (i\not{\partial} + m)\ i\,\Delta^+(y-x)$$

Thus,

$$i S_{F\,\alpha\beta}(x-y) = \theta(x^0-y^0)\langle 0|\, \psi_\alpha(x)\, \bar\psi_\beta(y)\,|0\rangle$$
$$- \theta(y^0-x^0)\,\langle 0|\, \bar\psi_\beta(y)\, \psi_\alpha(x)\,|0\rangle$$

$$= \theta(x^0-y^0)(i\cancel{\partial}+m)\; i\Delta^+(x-y)$$
$$+ \theta(y^0-x^0)\,(i\cancel{\partial}+m)\; i\Delta^+(y-x)$$

$$\Delta^+(y-x) = -\Delta^-(x-y)$$

$$= i\,(i\cancel{\partial}+m)\left[\theta(x^0-y^0)\,\Delta^+(x-y) - \theta(y^0-x^0)\,\Delta^-(x-y)\right]$$

$$= i\,(i\cancel{\partial}+m)\;\Delta_F(x-y) \qquad \text{Eq. (3.56a)}$$

$$= i\;(i\cancel{\partial}+m)\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \qquad \text{Eq. (3.58)}$$

Result

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{\cancel{p}+m}{p^2-m^2+i\epsilon}\, e^{-ip\cdot(x-y)}$$

Homework Problems due Friday March 25

Problem 2.
(a) Mandl and Shaw problem 4.3.
(b) Mandl and Shaw problem 4.4.

Problem 3.
Mandl and Shaw problem 4.5.