How is the Dirac equation consistent with special relativity? (Appendix)

$$(i\, \gamma \cdot \partial - m)\, \psi(\boldsymbol{x}) = 0 \qquad (1)$$

Now consider two inertial frames,

$\boldsymbol{x}^\mu = \{ x^0, x^1, x^2, x^3 \}$ ref. frame $\mathcal{F}$

$\boldsymbol{x}'^\mu = \{ x'^0, x'^1, x'^2, x'^3 \}$ ref. frame $\mathcal{F}'$

$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$ where $\Lambda^\mu{}_\nu$ = the Lorentz transformation matrix

Suppose $\psi(\boldsymbol{x})$ is a solution of the Dirac equation in the unprimed coordinates.
What is the solution in the primed coordinates?

I.e., we want

$$(i\, \gamma' \cdot \partial' - m)\, \psi'(\boldsymbol{x}') = 0 \qquad (2)$$

① *Question*: What is γ'^μ ?
Answer: $\gamma'^\mu = \gamma^\mu$.
Proof: Gamma matrices are the same in all inertial frames; e.g.,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

② *Question*: What is ∂'_μ ?
Answer: $\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu$
Proof: By the chain rule,

$$\frac{\partial Q}{\partial x'^\mu} = \frac{\partial Q}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} \quad \text{where } x^\nu = (\Lambda^{-1})^\nu{}_\rho\, x'^\rho$$

$$\partial'_\mu = (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu$$

Require $(i \ \gamma'.\partial' - m) \psi'(\boldsymbol{x'}) = 0$ (2)

③ Question: What is $\psi'(\boldsymbol{x'})$?
Answer:

$$\psi'(x') = S(\Lambda) \psi(x)$$ [Mandl&Shaw]

Here $S(\Lambda)$ is a 4 x 4 matrix;
it's the matrix such that Eq. (2) is satisfied.

Theorem 1

The matrix $S(\Lambda)$ must satisfy

$$S^{-1} \gamma^\mu S = \Lambda^\mu{}_\nu \ \gamma^\nu \quad .$$

Theorem 2

$$S(\Lambda) = \exp[-(i/2)\, \omega_{\mu\nu} \sigma^{\mu\nu}]$$

where $\sigma^{\mu\nu} = i/4 [\gamma^\mu , \gamma^\nu]$,

and $\omega_{\mu\nu}$ is antisymmetric w.r.t. exchange of μ and ν .

Proof of Theorem 1 ; by verification

$$
\begin{aligned}
& (i \ \gamma'.\partial' - m) \psi'(x') \\
&= (i \gamma^\mu \ (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m) S(\Lambda) \psi(x) \\
&= S S^{-1} (i \gamma^\mu \ (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m) S \psi(x) \\
&= S (i \underline{S^{-1} \gamma^\mu S} (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m) \psi(x) \\
&= S (i \Lambda^\mu{}_\lambda \ \gamma^\lambda (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m) \psi(x) \\
&= S (i \gamma^\lambda \partial_\lambda - m) \psi(x) \\
&= 0 .
\end{aligned}
$$

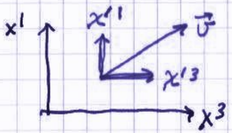
Q.E.D. (assuming S exists)

Theorem 2. What is S?

Consider a boost by velocity $\vec{v} = \beta\hat{n}$.

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -n^1\beta\gamma & -n^2\beta\gamma & -n^3\beta\gamma \\ -n^1\beta\gamma & 1+n^1n^1\rho & n^1n^2\rho & n^1n^3\rho \\ -n^2\beta\gamma & n^2n^1\rho & 1+n_2^2\rho & n^2n^3\rho \\ -n^3\beta\gamma & n^3n^1\rho & n^3n^2\rho & 1+n_3^2\rho \end{bmatrix}$$

where $\rho = \gamma - 1$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.



Check $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$x'^0 = \gamma x^0 - \beta\gamma\,\hat{n}\cdot\vec{x}$$
$$= \gamma x^0 - \gamma\,\vec{v}\cdot\vec{x} \quad ✓$$

$$x'^n = n^i \Lambda^i{}_\nu x^\nu$$
$$= n^i(-n^i\beta\gamma)x^0 + n^i\Lambda^i{}_j x^j$$

$$= -\beta\gamma x^0 + n^i[\delta^i_j + n^i n^j \rho]x^j$$
$$= -\beta\gamma x^0 + (1+\rho)\,\hat{n}\cdot\vec{x}$$
$$= -\beta\gamma\, x^0 + \gamma\, x^n \quad ✓$$

$$\hat{n}\times\vec{x}' = \hat{n}\times(\underbrace{\hat{e}_i\Lambda^i{}_\nu}_{\text{only } \delta^i_j \text{ is nonzero}} x^\nu) = \hat{n}\times\vec{x} \quad ✓$$

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$\sigma^{\mu\nu} = S^{\mu\nu}$

Guess

$$\omega_{\rho\sigma} = \begin{bmatrix} 0 & -\omega n^1 & -\omega n^2 & -\omega n^3 \\ \omega n_1 & 0 & 0 & 0 \\ \omega n^2 & 0 & 0 & 0 \\ \omega n^3 & 0 & 0 & 0 \end{bmatrix}$$

$$\omega\cdot S = -\omega n^i S^{0i} \times 2$$

$$S^{0i} = \frac{i}{4}[\gamma^0,\gamma^i] = \frac{i}{4}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix} - \begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]$$
$$= \frac{i}{2}\begin{pmatrix}0&\sigma^i\\\sigma^i&0\end{pmatrix}$$

$$\omega\cdot S = -i\omega\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix}$$

$$\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix}\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$S(\Lambda) = e^{-\frac{\omega}{2}\begin{pmatrix}0&\sigma_n\\\sigma_n&0\end{pmatrix}} = \begin{pmatrix}\cosh\frac{\omega}{2} & -\sigma_n\sinh\frac{\omega}{2} \\ -\sigma_n\sinh\frac{\omega}{2} & \cosh\frac{\omega}{2}\end{pmatrix}$$
$$= \begin{pmatrix}c_2 & -\sigma_n s_2 \\ -\sigma_n s_2 & c_2\end{pmatrix}$$

$$S^{-1}(\Lambda) = \begin{pmatrix}c_2 & \sigma_n s_2 \\ \sigma_n s_2 & c_2\end{pmatrix}$$

Theorem 2 To prove: $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$

First, consider $\mu = 0$.

$\underline{\mu = 0}$

$$\begin{pmatrix} c_2 & \sigma_n s_2 \\ \sigma_n s_2 & c_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_2 & -\sigma_n s_2 \\ -\sigma_n s_2 & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_2^2 + s_2^2 & -\sigma_n 2 c_2 s_2 \\ \sigma_n 2 c_2 s_2 & -c_2^2 - s_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} C & -\sigma_n S \\ \sigma_n S & -C \end{pmatrix} \qquad \begin{array}{l} C = \cosh\omega \\ S = \sinh\omega \end{array}$$

$$= \Lambda^0{}_\nu \gamma^\nu = \Lambda^0{}_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Lambda^0{}_i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda^0{}_0 & \sigma^i \Lambda^0{}_i \\ -\sigma^i \Lambda^0{}_i & -\Lambda^0{}_0 \end{pmatrix}$$

$$\therefore \; \Lambda^0{}_0 = \cosh\omega = \gamma$$

$$\& \; \Lambda^0{}_i = -n^i S = -n^i \sinh\omega = -n^i \beta\gamma$$

$$\gamma^2 - \beta^2\gamma^2 = 1 \text{ so } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Now consider $\mu = i$.

$\underline{\mu = i}$

$$\begin{pmatrix} c_2 & \sigma_n c_2 \\ \sigma_n c_2 & c_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} c_2 & -\sigma_n s_2 \\ -\sigma_n s_2 & c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -n^i S & \sigma^i + 2n^i \sigma_n s_2^2 \\ -\sigma^i - 2n^i \sigma_n s_2^2 & n^i S \end{pmatrix}$$

$$= \Lambda^i{}_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Lambda^i{}_j \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

Exercise:

$$\Lambda^i{}_0 = -n^i S \qquad (S = \sinh\omega) \; (\beta\gamma)$$

$$\Lambda^i{}_j = \delta_{ij} + 2 n^i n^j s_2^2$$

$$= \delta_{ij} + n^i n^j (C - 1) \; (\gamma - 1)$$

Q.E.D.

Example.

Consider a boost in the z direction.

The Lorentz transformation matrix is

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}}_{\Lambda^{\mu}{}_{\nu}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

What is $S(\Lambda)$?

$$S(\Lambda) = \exp\left\{ -\omega/2 \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \right\}$$

$$S(\Lambda) = \begin{pmatrix} \cosh(\omega/2) & -\sigma_3 \sinh(\omega/2) \\ -\sigma_3 \sinh(\omega/2) & \cosh(\omega/2) \end{pmatrix}$$

$$S(\Lambda) = \begin{pmatrix} \cosh(\omega/2) & -\sigma_3 \sinh(\omega/2) \\ -\sigma_3 \sinh(\omega/2) & \cosh(\omega/2) \end{pmatrix}$$

Example

The Dirac spinors for a particle at rest are

$$u_1'(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2'(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(Frame $\mathcal{F}'$ is the rest frame of the particle.)

Therefore the Dirac spinors for a particle with 3-momentum **p** = (0, 0, p^3) are

$$u_1(\vec{p}) = S^{-1}(\Lambda)\, u_1'(\vec{0})$$

$$= \begin{pmatrix} \mathbb{1} c_2 & \sigma_3 s_2 \\ \sigma_3 s_2 & \mathbb{1} c_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_2 \\ 0 \\ s_2 \\ 0 \end{pmatrix}$$

$$c_2 = \cosh\frac{\omega}{2} \quad \text{and} \quad \cosh\omega = \gamma = \frac{E}{mc^2}$$

$$c_2 = \sqrt{\tfrac{1}{2}(\gamma+1)} = \sqrt{\frac{E+m}{2m}}$$

$$s_2 = \sqrt{\frac{E-m}{2m}} \qquad (c_2^2 - s_2^2 = 1)$$

$$u_1(\vec{p}) = \frac{1}{\sqrt{2m}} \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix} = N \begin{pmatrix} E+m \\ 0 \\ p_z \\ 0 \end{pmatrix}$$

Sim., $u_2(\vec{p}) = N \begin{pmatrix} 0 \\ E+m \\ 0 \\ -p_z \end{pmatrix}$

agrees with the eigenvectors of $\boldsymbol{\gamma.p}$.

Dirac Field Bilinears

$\bar{\psi}\ \psi$ is a scalar

$\bar{\psi}\ \gamma^\mu\ \psi$ is a vector

$\bar{\psi}\ \gamma^\mu\ \gamma^\nu\ \psi$ is a tensor

$\bar{\psi}\ \gamma^5\ \psi$ is a pseudo-scalar

$\bar{\psi}\ \gamma^\mu\ \gamma^5\ \psi$ is a pseudo-vector

Proof

$$\bar{\psi}'\bar{\psi}' = \psi'^\dagger \gamma^0 \psi'$$

$$= \psi^\dagger S(\Lambda)^\dagger \gamma^0 S(\Lambda)\psi$$

$$= \psi^\dagger \gamma^0\gamma^0 S(\Lambda)^\dagger \gamma^0 S(\Lambda)\psi$$

where

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$\gamma^0 S(\Lambda)^\dagger \gamma^0 = \gamma^0 e^{\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger}\gamma^0$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$ because $\gamma^0 (S^{\mu\nu})^\dagger \gamma^0 = S^{\mu\nu}$

$$\gamma^0 S(\Lambda)^\dagger \gamma^0 S(\Lambda) = e^{\frac{i}{2}\omega\cdot S}\, e^{-\frac{i}{2}\omega\cdot S} = 1$$

$$\bar{\psi}'\psi' = \bar{\psi}\psi \quad \text{Q.E.D.}$$

Etc, similarly.