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Section 6.1
Natural dimensions and units

Set $\hbar = 1$ and $c = 1$.

I.e., omit all factors of $\hbar$ and c
during calculations.

At the end of the calculation, restore the factors of $\hbar$ and c by unit analysis.
(There is always a unique way to do this.)

Read Section 6.1 for a detailed discussion of this method.

Section 6.2
The S-matrix expansion

We already know this from PHY 855.

Use the Interaction Picture to calculate the transition matrix elements, treating the interaction in perturbation theory.

(1) The Hamiltonian is $H = H_0 + H_I$.

(2) The S-matrix, $S = U(\infty, -\infty)$.

The goal of quantum theory is to calculate *time evolution* of the states of the system.

Let $|\psi,t_0\rangle$ be the state at time t_0.
Then the state at time t is (in the Schroedinger picture)

$$|\psi,t\rangle = e^{-iH(t-t_0)}\,|\psi,t_0\rangle$$

Now suppose $t_0 \to -\infty$,
and $|\psi,t_0\rangle \to |\,i\,\rangle$,
where $|\,i\,\rangle$ consists of free particles (i.e., very far apart).
Then the state at time t, for $t \to \infty$, is

$$|\psi,t\rangle = e^{-iH(t-t_0)}\,|\,i\,\rangle$$
$$= \sum_j |\,j\,\rangle\langle\,j\,|\,e^{-iH(t-t0)}\,|\,i\,\rangle$$
$$= \sum_j |\,j\,\rangle\;S_{ji}$$

So the probability that the state $|\,f\,\rangle$ will be observed at time t is

$$P(\,i \to f\,) = |\,S_{fi}|^2\,.$$

Now, consider, in the Heisenberg picture,

$<\Psi_0|\ A(t)\ B(t_0)\ |\Psi_0>$

$= <\Psi_0\ |\ A(0)\ e^{-iHt}\ e^{iHt_0}\ B(0)\ |\Psi_0>$

$= <\Psi_0\ |\ A(0)\ U(t,t_0)\ B(0)\ |\Psi_0>$;

...which must be the same function in the interaction picture

$= < 0\ |\ A(0)$

$\int_{t0}^{t} T \exp\left[-i\, H_I(t')\, t'\right] dt' \quad B(0)\ |\ 0>_{IP}$

Thus,

$U(\infty,-\infty) = S =$

$$\int_{t0}^{t} T \exp\left[-i\, H_I(t')\, t'\right] dt'$$

(3)

$\Rightarrow$ Mandl&Shaw Eq. (6.22b)

$$S = \sum_{n=0}^{\infty} (-i)^n / n! \int dt_1 \ldots \int dt_n \; T\{ H_I(t_1) \ldots H_I(t_n) \} .$$

Now apply this result to Quantum Field Theory:

$$H_I(t) = \int d^3x\, \mathcal{H}_I(t,\mathbf{x})$$

$$S = \sum_{n=0}^{\infty} (-i)^n / n! \int d^4x_1 \ldots \int d^4x_n \; T\{ \mathcal{H}_I(\boldsymbol{x}_1) \ldots \mathcal{H}_I(\boldsymbol{x}_n) \} .$$

Eq. (6.23)

"This equation is the Dyson expansion of the S-matrix. It forms the starting point for the approach to perturbation theory used in this book."

Usually $\mathcal{H}_I(\boldsymbol{x}) = -\mathcal{L}_I(\boldsymbol{x})$

Section 6.3.
Wick's theorem

We studied this in PHY 855.

① The time-ordered product of two fields is equal to the normal-ordered product plus a c-number contraction.

$$T\{A(x_1)\, B(x_2)\} =$$

$$N\{A(x_1)\, B(x_2)\} + \underbrace{A(x_1)\, B(x_2)}$$

② The c-number contraction is a propagator.

$$\underbrace{A(x_1)\, B(x_2)} \equiv \langle 0 |\; T\{\, A(x_1)\, B(x_2)\,\}\; | 0 \rangle$$

③ Examples

Real scalar field (6.32a) :

$$\underbrace{\varphi(x_1)\varphi(x_2)} = i\, \Delta_F(x_1 - x_2)$$

Complex scalar field (6.32b):

$$\underbrace{\varphi(x_1)\varphi^{\dagger}(x_2)} = \underbrace{\varphi^{\dagger}(x_2)\, \varphi(x_1)} = i\, \Delta_F(x_1 - x_2)$$

Dirac field (6.32c):

$$\underbrace{\psi(x_1)\overline{\psi}(x_2)} = -\underbrace{\overline{\psi}(x_2)\psi(x_1)} = i\, S_F(x_1 - x_2)$$

Electromagnetic field (6.32d):

$$\underbrace{A_\mu(x_1)\, A_\nu(x_1)} = \text{(a future lecture)}$$

The contraction of two distinguishable fields is 0. *E.g.*, contraction of electron field and quark field = 0.

④ The time-ordered product of *any number of fields* can be written as the sum of all normal ordered products multiplied by c-number contractions. ***(Wick's Theorem)***

⑤ The vacuum expectation value of any normal-ordered product is 0.

⑥ The vacuum expectation value of any time-ordered product is the sum of all complete contractions.

That's where we get Feynman diagrams.

Exercise: Read Chapter 6.

Homework Problem due April 1.

Problem 4.
Use anticommutation relations and definitions to prove

$$T\{\psi_{\alpha}(\boldsymbol{x})\,\overline{\psi}_{\beta}(\boldsymbol{y})\} - N\{\psi_{\alpha}(\boldsymbol{x})\,\overline{\psi}_{\beta}(\boldsymbol{y})\} = i\, S_{F\,\alpha\beta}(\boldsymbol{x}-\boldsymbol{y})$$