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Section 5.1. The Classical Electromagnetic Fields

Maxwell's theory in covariant form

⚡ The field tensor $F^{\mu\nu}(x)$

$$\begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} = F^{\mu\nu}$$

$$F^{0i} = -F^{i0} = E_i$$
$$F^{ij} = \varepsilon_{ijk} B_k$$

⚡ The field equations

$$\partial_\nu F^{\mu\nu} = s^\mu \,; \qquad \text{which requires } \partial_\mu s^\mu = 0\,; \qquad \textbf{(CM1)}$$

conservation of charge

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0\,. \qquad \textbf{(CM2)}$$

$\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} + \partial \vec{B}/\partial t = 0$

⚡ The 4-vector potential $A^\mu(x)$

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

$$\boxed{\begin{aligned} F^{0i} &= \partial^i A^0 - \partial^0 A^i \\ \mathbf{E} &= -\nabla\Phi - \partial\mathbf{A}/\partial t \\ F^{ij} &= \partial^j A^i - \partial^i A^j \\ &= -\partial_j A^i + \partial_i A^j = \varepsilon_{ijk}(\nabla\times\mathbf{A})_k \end{aligned}}$$

This makes (CM2) = 0 automatically, i.e., for any $A^\mu(x)$. Now (CM1) becomes

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = s^\mu$$

⚡ The classical gauge transformation

$F^{\mu\nu}$ is invariant under the transformation

$$A^\mu(x) \longrightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu f(x)$$

for any scalar function $f(x)$.

Proof:

$$\boxed{\begin{aligned} F'^{\mu\nu} &= \partial^\nu A'^\mu - \partial^\mu A'^\nu \\ &= \partial^\nu(A^\mu + \partial^\mu f) - \partial^\mu(A^\nu + \partial^\nu f) = F^{\mu\nu} \end{aligned}}$$

▌Lagrangian dynamics for electromagnetism

⚡ One classical Lagrangian density

Let's consider

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - s_\mu A^\mu \qquad \textbf{(L1)}$$

Lorentz invariant? □

Gauge invariant? χ *But* $\delta L = \partial^\mu(-s_\mu f)$

Gauge invariant? □

Then what is the field equation for A^μ ?

Lagrange's equation:

$$\boxed{\begin{aligned} &\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad \text{where } \phi = A_\rho \\ &= \partial_\lambda\Big(-\tfrac{1}{2}F^{\mu\nu}\underbrace{\frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\rho)}}_{=\,\delta_{\lambda\nu}\delta_{\rho\mu} - \delta_{\lambda\mu}\delta_{\rho\nu}}\Big) + s^\rho \\ &= -\partial_\nu F^{\rho\nu} + s^\rho = 0 \quad \text{(Maxwell)} \end{aligned}}$$

However, the Lagrangian density (**L1**) is not compatible with canonical quantization.

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - s_\mu A^\mu \qquad (\mathbf{L1})$$

Canonical momentum (a 4-vector)

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} \quad \text{and} \quad [\phi, \pi_\phi] = i\hbar\, \delta^3(x-y)$$

For $\phi = A_\rho \Rightarrow \pi^\rho = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_\rho)}$ $/\rho = 0123/$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^i)} = -\frac{1}{2} F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial(\partial_0 A_i)}$$

$$= -\frac{1}{2} F^{\mu\nu} \{ \delta_{0\nu}\delta_{i\mu} - \delta_{0\mu}\delta_{i\nu} \}$$

$$= -F^{i0} = E^i \quad \underline{(ok)}$$

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0 ;$$

So canonical quantization fails,

$[A^0, \pi^0] \neq i\hbar\, \delta^3(x-y)$.

⚡ Another classical Lagrangian density (Fermi)

$$\mathfrak{L} = -\tfrac{1}{2} (\partial_\nu A_\mu)(\partial^\nu A^\mu) - s_\mu A^\mu \qquad (\mathbf{L2})$$

✓First check the field equation:

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\rho)} \right) - \frac{\partial \mathcal{L}}{\partial A_\rho} = \partial_\lambda(-\partial^\lambda A^\rho) + s^\rho$$

$$\Box A^\rho = s^\rho$$

OK, provided A^ρ obeys the Lorentz gauge condition, $\partial_\rho A^\rho = 0$.

✓Then check the compatibility with canonical quantization:

$$\pi^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\rho)} = -\partial^0 A^\rho = -\dot A^\rho \neq 0$$

OK, but now there are 4 degrees of freedom:

Trans$_1$, Trans$_2$, Long , A0. *<u>L and A0 are unphysical.</u>*

The Lorentz gauge condition in the classical theory.

In the classical theory, suppose we have fields (i.e., the tensor $F^{\mu\nu}(\boldsymbol{x})$) with a 4-vector potential $A^{\mu}(\boldsymbol{x})$.

But suppose $\partial_{\mu}A^{\mu} \neq 0$.

There exists a gauge transformation $A^{\mu}(\boldsymbol{x}) \longrightarrow A'^{\mu}(\boldsymbol{x})$ such that $\partial_{\mu}A'^{\mu} = 0$.

Proof:

Let $\quad f(\boldsymbol{x}) = -\Box^{-1}(\partial_{\rho}A^{\rho})$

The gauge transformation does not change the fields, $F^{\mu\nu}(\boldsymbol{x})$.

The Lorentz gauge versus the Coulomb gauge.

$\partial_{\mu}A^{\mu} = 0$

versus

$\nabla \cdot \mathbf{A} = 0$ and $-\nabla^2\Phi = j_0$

The theory is gauge invariant;
i.e., the physical predictions are the same for either gauge condition.

The Coulomb gauge has an advantage: it is a "unitary gauge". But it has a disadvantage: it is not manifestly Lorentz invariant.

The Lorentz gauge has an advantage: it is manifestly Lorentz invariant. But is has a disadvantage: it has unphysical degrees of freedom.

Section 5.2: ① Use the "Gupta-Bleuler formalism" to impose the condition $\partial_{\mu}A^{\mu} = 0$; and ② don't worry about it.

Section 13.4.
Gauge Independent Quantization?

"… … …

As in the canonical formulation, the electromagnetic field cannot be consistently quantized using path integrals without 'fixing a gauge.'

"

Section 14.1. Gluon Fields

"

Section 14.1.5.

The electromagnetic field revisited.

It will be instructive to comment briefly on the result of applying the *Faddeev-Popov procedure* to the electromagnetic field.

"

But now going back to Section 5.1:

Plane wave solutions

The free field theory ($s^\mu(x) = 0$) has just

$$\Box A^\mu = 0 \,. \qquad \textit{(impose LG condition later)}$$

The plane wave solutions are

$$A^\mu(x) = \varepsilon_r^{\;\mu}(\mathbf{k})\, e^{-i\,k.x} \quad \text{w/} \quad r \in \{0, 1, 2, 3\}$$

where $k^0 = \pm|\mathbf{k}|$.

The ***four*** polarization vectors are normalized in some way; $\varepsilon_1^{\;\mu}$ and $\varepsilon_2^{\;\mu}$ are transverse w. r. t. **k**, $\varepsilon_3^{\;\mu}$ is longitudinal, and $\varepsilon_0^{\;\mu}$ is timelike.

The *general* solution is

$$A^\mu(x) = \sum_{\mathbf{k}} \sum_r \left(\frac{\hbar c^2}{2\Omega\omega}\right)^{1/2} \epsilon_r^\mu(\mathbf{k}) \left\{ a_r(\mathbf{k})\, e^{-ik\cdot x} + a_r^*(\mathbf{k})\, e^{ik\cdot x} \right\}$$