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SECTION 5.2. COVARIANT QUANTIZATION

Review the covariant equations for the classical electromagnetic field

$F = \text{curl } A$; $\quad F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$

In the Lorentz gauge ($\partial_\mu A^\mu = 0$) we can use the Lagrangian density

$$L = -\tfrac{1}{2}\,(\partial_\nu A_\mu)(\partial^\nu A^\mu) - s_\mu A^\mu$$

Now apply canonical quantization to this Lagrangian density. Then calculate

$$[\, A^\mu(x)\, ,\, A^\nu(y)\,] = i\, D^{\mu\nu}\,(x-y)$$

and

$$\langle 0|T\, A^\mu(x)\;\; A^\nu(y)\, |0\rangle = i\, D_F^{\mu\nu}\,(x-y)$$

$L = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) - s_\mu A^\mu$

Recall the real scalar field,

$$L_\varphi = \tfrac{1}{2}(\partial_\nu\varphi)(\partial^\nu\varphi) - \tfrac{1}{2} m^2 \varphi^2$$

$$[\varphi(x),\varphi(y)] = i\,\Delta(x-y)$$

$$\langle 0|\, T\, \varphi(x)\, \varphi(y)\, |0\rangle = i\,\Delta_F(x-y)$$

Note that the free e.m. field has

$$L = -\tfrac{1}{2}(\partial_\nu A^0)(\partial^\nu A^0) + \tfrac{1}{2}(\partial_\nu A^i)(\partial^\nu A^i)$$

(sum i = 1 2 3)

- $A^i(x)$ is just like $\varphi(x)$ *with m=0*

$\therefore \quad [A^i(x),A^i(y)] = i\,\Delta(x-y)$

$\therefore \quad \langle 0|\, T\, A^i(x)\, A^i(y)\, |0\rangle = i\,\Delta_F(x-y)$

(no sum on i)

Different i and j are independent

$\therefore \quad [A^i(x),A^j(y)] = i\,\delta_{ij}\,\Delta(x-y)$

$\& \quad \langle 0|\, T\, A^i(x)\, A^j(y)\, |0\rangle = i\,\delta_{ij}\,\Delta_F(x-y)$

- $A^0(x)$ is a little different (the sign)

$$\Pi^0 = \partial L \,/\, \partial(\partial A^0/\partial t) = -\,\partial A^0/\partial t$$

(compare $\Pi_\varphi = \partial\varphi/\partial t$)

So the commutator is

$$[\,A^0(x)\,,\,A^0(y)\,] = -i\,\Delta(x-y)$$

<u>Commutator result</u>

$[\,A^\mu(x)\,,\,A^\nu(y)\,] = -i\,g^{\mu\nu}\,\Delta(x-y)$

$\qquad\qquad = i\,D^{\mu\nu}(x-y)$

$\therefore \quad D^{\mu\nu}(x-y) = -\,g^{\mu\nu}\,\Delta(x-y)$

(with m = 0)

- As for the propagator,

same thing $\Rightarrow$

$$<0|\, T\, A^\mu(x)\, A^\nu(y)\, |0> = -i\, g^{\mu\nu}\, \Delta_F(x-y)$$

$$= i\, D_F^{\mu\nu}(x-y)$$

$$D_F^{\mu\nu}(x-y) = -\, g^{\mu\nu}\, \Delta_F(x-y)$$

(with m = 0)

The Fourier integral,

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} D_F^{\mu\nu}(k)\, e^{-i\,k.x}$$

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2+i\,\varepsilon}$$

Expansion in plane waves

$$A^\mu(x) = \sum_{\mathbf{k}} \sum_{r=0}^{3} \left(\frac{\hbar c^2}{2\Omega\omega}\right)^{1/2} \epsilon_r^\mu(\vec{k}) \left\{ a_r(\mathbf{k})\, e^{-ik\cdot x} + a_r^\dagger(\mathbf{k})\, e^{ik\cdot x} \right\}$$

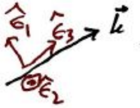
where $k^0 = |\mathbf{k}| = \omega$.

The four-vector polarization vectors are defined like this:

$\epsilon_0^\mu(\mathbf{k}) = \eta^\mu \equiv (1,0,0,0)$ pure time like

$\epsilon_i^\mu(\mathbf{k}) = (0, \hat{\epsilon}_i(\mathbf{k}))$ pure space like $(i=1\,2\,3)$

$\hat{\epsilon}_3(\mathbf{k}) = \mathbf{k}/\omega$ longitudinal



$\underbrace{\hat{\epsilon}_1, \hat{\epsilon}_2}_{\text{transverse}}, \hat{\epsilon}_3$ form an orthogonal triad.

The canonical commutation relation

$[\, A^\mu(x)\, ,\, A^\nu(y)\,] = -i\, g^{\mu\nu}\, \Delta(x-y)$ implies

$$[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')] = \delta_{rs}\, \delta_{\mathbf{kk}'}\, \zeta_r \quad \text{where } \zeta_r = \begin{cases} -1 \text{ for } 0 = r \\ +1 \text{ for } 123 \end{cases}$$

The constraint $\partial_\mu A^\mu = 0$;
i.e., Lorentz gauge quantization

⊙ We cannot set $\partial_\mu A^\mu = 0$ as an operator equation.

Proof

$$[\partial_\mu A^\mu(x), A^\nu(y)] = -i\,\partial^\nu \Delta(x-y)$$

and it is not 0.

⊙ The Gupta-Bleuler *formalism*:

(Comment: it’s not a theory;
it’s not a model; it’s a *formalism*.)

Apply the constraint, $\partial_\mu A^\mu = 0$, to the states of the Hilbert space; require

$$\partial_\mu A^{(+)\mu}\,|\Psi\rangle = 0\ ,$$

for any physical state $|\Psi\rangle$.

⊙ Theorem

$[\, a_3(\mathbf{k}) - a_0(\mathbf{k})\,]\ |\Psi\rangle = 0$ for all $\mathbf{k}$

Proof

$$\partial_\mu A^{(+)\mu} = \partial_\mu \sum_{\vec{k}} \sum_{r=0}^{3} \frac{1}{\sqrt{2\Omega\omega}}\, \epsilon_r^\mu(\vec{k})\, a_r(\vec{k})\, e^{-ik\cdot x}$$

$$= \sum_{\vec{k}} \sum_{r} \frac{1}{\sqrt{2\Omega\omega}} (-ik_\mu)\, \epsilon_r^\mu(\vec{k})\, a_r(\vec{k})\, e^{-ik\cdot x}$$

$$= \sum_{\vec{k}} \sum_{r=0,3} \frac{(-i)}{\sqrt{2\Omega\omega}} \left[k^0 \epsilon_r^0 - \vec{k}\cdot\vec{\epsilon}_r\right] a_r(\vec{k})\, e^{-ik\cdot x}$$

$$= \sum_{\vec{k}} \frac{(-i)}{\sqrt{2\Omega\omega}} \underbrace{\left\{ k^0 a_0(\vec{k}) - \vec{k}\cdot\vec{\epsilon}_3\, a_3(\vec{k}) \right\}}_{=\,\omega\,(a_0(\vec{k}) - a_3(\vec{k}))} e^{-ik\cdot x}$$

$\partial_\mu A^{(+)\mu}(x)|\Psi\rangle = 0$ implies

$$[\,a_0(\vec{k}) - a_3(\vec{k})\,]\ |\Psi\rangle = 0 \quad \text{all } \vec{k}$$

$[\ a_3(\mathbf{k}) - a_0(\mathbf{k})\ ]\ |\Psi\rangle = 0,$ (★)

for any physical state; arb $\mathbf{k}$.

What does it mean?

⊙ Recall the free vacuum $|0\rangle$; it has

$$a_r(\mathbf{k})\ |0\rangle = 0 \quad \text{for} \quad r = 0\ 1\ 2\ 3$$

so $|0\rangle$ obeys (★)

⊙ Now consider $a^{\dagger}_{\mathbf{r}}(\mathbf{k})\ |0\rangle$.

For r = 1, 2, it obeys (★).
Creating any transverse photons will yield a physical state.

For r = 3, 0, the state does not obey (★).
Creating single longitudinal or scalar photons will yield an unphysical state; or, a physical state requires creating longitudinal and scalar together.

⊙ Example. Consider

$$|\psi\rangle = [\ a^{\dagger}_{\mathbf{3}}(\mathbf{p}) - a^{\dagger}_{\mathbf{0}}(\mathbf{q})\ ]\ |0\rangle$$

It has

$$a_{\mathbf{3}}(\mathbf{k})\ |\psi\rangle = \delta_{\mathbf{k},\mathbf{p}}\ |0\rangle$$

$$a_{\mathbf{0}}(\mathbf{k})\ |\psi\rangle = \delta_{\mathbf{k},\mathbf{q}}\ |0\rangle$$

So $|\psi\rangle$ is a physical state (★) provided that $\mathbf{p} = \mathbf{q}$.

(M. & Sh. problem 5.2; homework)

⊙ But that is just a gauge transformation.

(M. & Sh. problem 5.3; homework)

Results of the formalism

In the Gupta-Bleuler formalism,

- we only calculate transition amplitudes between states with transverse photons;
- i.e., longitudinal and scalar photons are not included in asymptotic states;
- but longitudinal and scalar photons do exist as virtual particles;
- i.e., we use the full propagator $-g^{\mu\nu}/(k^2+i\varepsilon)$.

This is all we need to proceed.

So now we could forget about the Gupta-Bleuler formalism.

But note what Mandl and Shaw say at the end of Section 5.2:

"...
For most purposes, the complete formalism is not required.
[footnote: if interested, read 3 other books.]
"

An alternative approach (which is required for QCD but optional for QED) is to use ***functional integration*** with the ***Faddeev-Popov formalism***. (M.&Sh. chapters 10 - 14)