Wednesday March 30

SECTION 5.3
THE PHOTON PROPAGATOR

(We could skip this, but there is something interesting here; interesting for the theory, but not really useful for applications of the theory.)

The Lorentz gauge ($\partial_\mu A^\mu = 0$) has a covariant photon propagator

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} D_F^{\mu\nu}(k)\, e^{-ik\cdot x}$$

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2+i\epsilon}$$

But it comes from the Gupta-Bleuler formalism, which seems abstract and unphysical.

The Coulomb gauge
($\nabla \cdot \mathbf{A} = 0$ and $\Phi = -\nabla^{-2} j^0$)
does not have unphysical "longitudinal and scalar photons"; but it is hard to use because the propagator is complicated and non-covariant.

Are the two gauge choices consistent with each other?

Yes, because the physical predictions are the same for the two methods.

How can that be?
The propagators are very different. But propagators are not gauge invariant.
All physical predictions are gauge invariant.

__Prove that they are gauge equivalent__

The crucial equation is equation (5.40), which is just a mathematical identity satisfied by the Lorentz gauge propagator…

$$D_F^{\mu\nu}(k) = \frac{1}{k^2+i\epsilon}\sum_{r=0}^{3} \epsilon_r^{\mu}(\vec{k})\,\epsilon_r^{\nu}(\vec{k})\,\zeta_r$$

$$(\zeta_0 = -1\,;\ \zeta_1 = +1)$$

$$\begin{aligned}
&= \frac{1}{k^2+i\epsilon}\Big\{ \sum_{r=1}^{2} \epsilon_r^{\mu}(\vec{k})\,\epsilon_r^{\nu}(\vec{k}) && \text{transverse} \\
&\qquad + \frac{(k^\mu - k\cdot n\, n^\mu)(k^\nu - k\cdot n\, n^\nu)}{(k\cdot n)^2 - k^2} && \text{longitudinal} \\
&\qquad - n^\mu n^\nu \Big\} && \text{scalar} \\
& && n^\mu = (1,0,0,0) \\
&= {}_T D_F^{\mu\nu} + {}_C D_F^{\mu\nu} + {}_R D_F^{\mu\nu}
\end{aligned}$$

… where …

where

$$ {}_{C}D_F^{\mu\nu} = \frac{1}{k^2}\left[\frac{(k\cdot n)^2}{(k\cdot n)^2 - k^2} - 1\right] n^\mu n^\nu $$

$$ = \frac{n^\mu n^\nu}{(k\cdot n)^2 - k^2} $$

and ${}_{R}D_F^{\mu\nu}$ = the remainder

${}_{C}D_F^{\mu\nu}(k)$ is the instantaneous Coulomb interaction Fourier transformed

$$ {}_{C}D_F^{\mu\nu}(k) = \frac{g^{\mu 0} g^{\nu 0}}{(k^0)^2 - k^2} = \frac{g^{\mu 0} g^{\nu 0}}{(\vec{k})^2} $$

$$ \int \frac{d^4k}{(2\pi)^4}\, {}_{C}D_F^{\mu\nu}(k)\, e^{-ik\cdot(x-y)} $$

$$ = \int \frac{dk^0}{2\pi} e^{-ik^0(x^0-y^0)} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{g^{\mu 0} g^{\nu 0}}{\vec{k}^2} $$

$$ = \frac{\delta(x^0-y^0)}{4\pi|\vec{x}-\vec{y}|}\, g^{\mu 0} g^{\nu 0} $$

Exchange of longitudinal and scalar photons creates the Coulomb interaction.

The Remainder

$$ {}_{R}D_F^{\mu\nu}(k) = k^\mu k^\nu d_1 + k^\mu n^\nu d_2 + n^\mu k^\nu d_3 $$

Any transition matrix element that depends on $D_F^{\mu\nu}$ will involve the operator

$$\int d^4x \int d^4y \quad S_{1\mu}(x)\, D_F^{\mu\nu}(x-y)\, S_{2\nu}(y)$$

where $\partial_\mu S_1^\mu = 0$ and $\partial_\mu S_2^\mu = 0$ (conservation of charge)

$$\int d^4x\, d^4y \quad S_{1\mu}(x)\; {}_R D_F^{\mu\nu}(x-y)\, S_{2\nu}(y)$$

$$= \int d^4x\, d^4y\, S_{1\mu}(x) \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \left[k^\mu k^\nu d_1 + k^\mu n^\nu d_2 + n^\mu k^\nu d_3 \right] S_{2\nu}(y)$$

$$= \int \frac{d^4k}{(2\pi)^4} \quad \hat{S}_{1\mu}(k) \left[k^\mu k^\nu d_1 + k^\mu n^\nu d_2 + n^\mu k^\nu d_3 \right] \hat{S}_{2\nu}(-k)$$

$\partial_\mu S_1^\mu(x) = 0$ implies $k_\mu \hat{S}_1^\mu(k) = 0$

also $k_\mu \hat{S}_2^\mu(-k) = 0$

So physical contributions of ${}_R D_F^{\mu\nu}$ are 0.

SECTION 7.1
FEYNMAN DIAGRAMS IN COORDINATE SPACE

To calculate transition probabilities, we need the S-matrix,

$$S_{fi} = \langle f|\ T \exp i \int d^4x\, \mathcal{L}_I(x)\ |i\rangle$$

where $|i\rangle$ and $|f\rangle$ are suitably normalized free particle states.
$(S_{fi} = \delta_{fi} + \Delta_{fi})$

To derive Feynman rules, we'll consider QED. Generalization to other field theories will be "obvious".

QED

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_I$$

$$\mathcal{L}_\psi = \bar{\psi}\,(i\gamma\cdot\partial - m)\,\psi$$

$$\mathcal{L}_A = -\tfrac{1}{2}\,(\partial_\nu A_\mu)(\partial^\nu A^\mu) \qquad (\text{w/ } \partial_\mu A^\mu = 0)$$

$$\mathcal{L}_I = e\,\bar{\psi}\,\gamma_\mu\,\psi\,A^\mu$$

$$\mathcal{L}_\psi + \mathcal{L}_I = \bar{\psi}\,(\underbrace{i\not\partial + e\not A}_{\text{"minimal coupling"}} - m)\,\psi$$

$$\mathcal{L}_A + \mathcal{L}_I = -\tfrac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) - J_\mu A^\mu$$

$$J_\mu = -e\,\bar{\psi}\gamma_\mu\psi$$

$$= \text{current density}$$

Consider a second-order contribution to S_{fi}

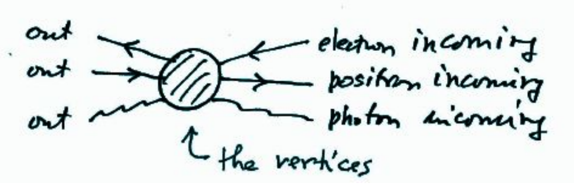
$$S^{(2)}_{fi} = \frac{i^2}{2!}\iint \langle f|\,T\,\mathcal{L}_I(x)\,\mathcal{L}_I(y)\,|i\rangle\, d^4x\, d^4y$$

$$= \frac{1}{2!}\iint \langle f|\,T\, ie\bar{\psi}\gamma_\mu\psi A^\mu(x)\; ie\bar{\psi}\gamma_\nu\psi A^\nu(y)\,|i\rangle\, d^4x\, d^4y$$

$$= \text{sum of complete contractions}$$

A Feynman diagram consists of :

- vertices;
- external electron lines and internal electron lines;
- external photon lines and internal photon lines.

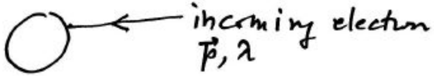


- Vertices:

Associated factor = i e γ_μ

- External electron lines:

Suppose |i> has an electron e(**p**,λ).
That must be annihilated by either $\psi(\boldsymbol{x})$ or $\psi(\boldsymbol{y})$.



$$\psi(x) = \sum_{\vec{p}} \sum_{\lambda} \left(\frac{2m}{2\Omega E}\right)^{1/2} \left[c_\lambda(\vec{p})\, u_\lambda(\vec{p})\, e^{-ip\cdot x} + d_\lambda^+(\vec{p})\, v_\lambda(\vec{p})\, e^{ip\cdot x} \right]$$

So the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} u_\lambda(\vec{p})\, e^{-ip\cdot x} \text{ or } e^{-ip\cdot y}$$

For a positron ***in | f >*** the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} v_{\lambda'}(\vec{p}\,')\, e^{ip'\cdot x}$$

$$\langle f | T\; ie\bar{\psi}\gamma_\mu \psi A^\mu(x)\; ie\bar{\psi}\gamma_\nu \psi A^\nu(y) | i \rangle$$

Suppose |f> has an electron e(**p′**,λ').
That must be created by either $\bar{\psi}(x)$ or $\bar{\psi}(y)$.

$$\bar{\psi}(x) = \sum_{\vec{p}} \sum_{\lambda} \left(\frac{2m}{2\Omega E}\right)^{1/2} \left[c_\lambda^+(\vec{p})\, \bar{u}_\lambda(\vec{p})\, e^{ip\cdot x} + d_\lambda(\vec{p})\, \bar{v}_\lambda(\vec{p})\, e^{-ip\cdot x} \right]$$

So the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} \bar{u}_{\lambda'}(\vec{p}\,')\, e^{ip'\cdot x} \text{ or } e^{ip'\cdot y}$$

For a positron ***in | i >*** the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} \bar{v}_\lambda(\vec{p})\, e^{-ip\cdot x} \text{ or } e^{-ip\cdot y}$$

- External photon lines:

Suppose | i > has a photon $\gamma(\mathbf{k},r)$;
r = 1 or 2 only .
That must be annihilated by either $A^\mu(\boldsymbol{x})$ or $A^\nu(\boldsymbol{y})$.

$$A^\mu(x) = \sum_{\vec{k}} \sum_{r=0}^{3} \left(\frac{1}{2\omega\Omega}\right)^{1/2} \epsilon_r^\mu(\vec{k}) \left(a_r(\vec{k})\, e^{-ik\cdot x} + a_r^\dagger(\vec{k})\, e^{ik\cdot x} \right)$$

So the associated factor is

$$\frac{1}{\sqrt{2\Omega\omega}}\, \epsilon_r^{\mu \text{ or } \nu}(\vec{k})\; e^{-ik\cdot x} \quad \text{or} \quad e^{-ik\cdot y}$$

Suppose | f > has a photon $\gamma(\mathbf{k}',r')$.
That must be created by either $A^\mu(\boldsymbol{x})$ or $A^\nu(\boldsymbol{y})$.

Then the associated factor is

$$\frac{1}{\sqrt{2\Omega\omega}}\, \epsilon_r^{\mu \text{ or } \nu}(\vec{k}')\; e^{ik'\cdot x} \quad \text{or} \quad e^{ik'\cdot y}$$

An incoming line has a factor of exp(-iq. x) and and outgoing line has a factor of exp(+iq.x), where $\hbar q^\mu$ is the 4-momentum.

▸ Internal electron lines:

Suppose Wick's theorem requires the contraction $\psi(x)\ \bar{\psi}(y)$

Then the associated factor is

$$S_F(x-y) =$$

$$= (2\pi)^{-4}\int\ d^4p\ \ S_F(p)\ \ e^{-ip.(x-y)}$$

▸ Internal photon lines:

Suppose Wick's theorem requires the contraction $A^\mu(x)\ A^\nu(y)$

Then the associated factor is

$$D_F{}^{\mu\nu}(x-y) =$$

$$= (2\pi)^{-4}\int\ d^4k\ \ D_F{}^{\mu\nu}(k)\ \ e^{-ik.(x-y)}$$

Example.

Electron-electron scattering,

$$e(p_1) + e(p_2) \rightarrow e(p_3) + e(p_4)$$

$\Rightarrow$ the Mott cross section;

Start here next time.

Do problem 7.1 in Mandl and Shaw.